Quantum measurements are crucial tools for obtaining information about quantum states. In the theory of quantum information, the measurement operators are represented by positive operator-valued measures (POVMs) – collections of positive semidefinite operators that sum up to the identity operator. Especially useful are informationally complete measurements, which allow to perform a full tomography (gain a complete knowledge) of a quantum state. Among the most popular highly-symmetric measurement operators, there are two classes of projective measurements. Symmetric, informationally complete (SIC) POVMs consist in such elements that the Hilbert-Schmidt (operator) inner product between different pairs always returns the same value. Mutually unbiased bases (MUBs) correspond to maximal sets of orthogonal projectors, where the transition probability between any two projectors onto vectors from different bases is constant. Interestingly, SIC POVMs and MUBs are often analyzed in the same context. They are both examples of complex projective 2-designs, correspond to affine planes but with the roles of points and lines interchanged, are operationally linked by joint measurability, and find important applications in the same problems of quantum information processing tasks. Examples of their empirical implementations include quantum state estimation, quantum key distribution, quantum entanglement detection, quantum communication protocols, and quantum teleportation. All this indicates that these two mathematical objects are closely related.

The goal of this project is to define, analyze, and construct measurement operators with weaker symmetry constraints but analogical properties and applications. In the preliminary results, it is shown that SIC POVMs and MUBs are special cases of a more general concept: equinumerous collections of mutually unbiased symmetric measurements that form an informationally complete set. Further analysis reveals two additional classes of measurement operators in every finite dimension. Interestingly, there exist choices of parameters for which the measurement operators are projectors but never of rank one. In the next step, collections of unequinumerous POVMs are introduced, where non-trivial families of measurement operators are already observed for qubits. Surprisingly, informationally complete sets of unequinumerous POVMs form conical two-designs (a generalization of complex projective two-designs to non-projective operators) only under very strict conditions. In this project, alternative generalizations of symmetric measurements are going to be considered. By relaxing different constraints, like informational completeness or equal trace of elements, it becomes possible to define families of operators that obey less restrictive symmetry conditions. This can reveal special algebraic properties that are otherwise unobservable. Moreover, initial research suggests that less symmetric measurements can potentially find just as many theoretical and experimental applications as highly symmetrized SIC POVMs, MUBs, and their non-projective generalizations.

A secondary objective of this project concerns the problems of whether there exist complete sets of MUBs and SIC POVMs in any finite dimensions. Finding additional analogues between their structures could be helpful in establishing a link between their explicit constructions. Recall that SIC POVMs exist in composite dimensions, which is undetermined for MUBs. On the other hand, there are known construction methods for maximal sets of MUBs in any prime and power prime dimensions, which is currently unobtainable for SIC POVMs. Therefore, the first step is to determine the existence of seven MUBs in dimension six and of SIC POVMs in a (preferably infinite) sequence of dimensions. These are popular open problems among both mathematicians and physicists due to being simple to formulate but technically non-trivial.

During my research, I am planning to implement mathematical tools from linear algebra, functional analysis, and representation theory to provide a satisfactory description of measurement operators and their properties. While considering their possible applications in quantum information, I am going to use advanced methods of theoretical and mathematical physics. From this project, I expect to open new research directions, as well as formulate new theorems and conjectures.