Standing Waves Under Constraints in Nonlinear Partial Differential Equations

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The research project is devoted to the study of various nonlinear stationary elliptic partial differential equations under additional constraints in the Lebesgue space of square-integrable functions. In particular, we focus on nonlinear Schrödinger equations and systems of coupled equations, which are among the most prominent nonlinear PDEs in the literature and arise naturally in numerous physical contexts, such as Bose–Einstein condensation, nonlinear optics, water waves, Langmuir waves in plasmas, and many others.

It is well known that the mass in the time-dependent Schrödinger equation is conserved, which naturally leads to the study of standing waves with a prescribed L^2-norm (or more generally, a constraint in a Lebesgue space). This constraint has a clear physical interpretation depending on the model: it represents, for example, the total number of atoms in a Bose–Einstein condensate or the power supply in nonlinear optical systems.

While some important open problems have been resolved in recent years, new mathematical challenges continue to emerge – many of which are closely tied to the physical relevance of the models. This project is centered around the following key directions:

- Normalized ground and bound states for nonlinear Schrödinger equations,
- Systems of coupled elliptic equations and their normalized solutions,
- Normalized problems in electrostatic Born–Infeld theory arising in nonlinear electromagnetism,
- Orthonormal solutions to elliptic equations modeling fermionic systems such as electrons, neutrons, or protons,
- Normalized solutions on bounded domains, inspired by models of CW laser beam propagation in hollow-core optical fibers.

Our research approach combines tools from several areas of mathematical analysis. In particular, we apply and further develop the following techniques: spectral theory of elliptic operators, elliptic regularity theory, variational methods, the mountain pass lemma, and topological degree theory.

In summary, the project lies at the intersection of several mathematical disciplines, including variational methods, partial differential equations, functional and spectral analysis, topological methods, and mathematical physics. The anticipated results aim to improve the understanding of complex physical models—such as those in nonlinear optics and quantum condensates—and to support the development of more efficient numerical methods for the associated equations. We expect to prove new existence results for constrained solutions and to introduce new variational techniques. Furthermore, the methods developed in this project may also be applied to a broader class of nonlinear PDEs, including nonlinear wave equations, systems of competing species modeling interspecific interactions, and nonlocal problems. Finally, the results obtained here are expected to contribute to further studies of the dynamics of time-dependent evolution equations.