The goal of the project is to investigate the connections between topology and dynamics on the combinatorial level in order to significantly strengthen the tools of topological data analysis on the one hand and provide methods for automated numerical analysis in dynamics on the other hand. To explain our motivation and aims we need to recall some facts from the history of mathematics together with their applicability and deficiencies in the relation to the current needs of sciences and technology.

The discovery of infinitesimal calculus by Leibniz and Newton in the second half of 17th century enabled the formulation of many problems in dynamics in the form of differential equations induced by vector fields. Unfortunately, in general such equations cannot be solved analytically. Although numerical methods provide approximate solutions over finite time intervals, they cannot be used to predict the long time (asymptotic) behavior of the system. As shown by Henri Poincaré at the end of the 19th century, this, in particular, applies to the equations governing the celestial mechanics. Looking for a remedy Poincaré invented what is today known as topology, a branch of mathematics studying the shape of geometric objects. The strong connections between topology and dynamics are particularly evident in Morse Theory, a fundamental theory in differential topology constructed in the first half of the 20th century by Marston Morse. In Morse Theory one studies a smooth real-valued function on a differentiable manifold. On the one hand this function reveals the geometric features of the manifold via sublevel sets of the function. On the other hand, it generates a gradient vector field and a flow on the manifold. Morse Theory captures the ties between the geometry of the manifold and the dynamics of the flow.

On the theoretical side Morse theory is useful both in topology and dynamics but it does not lead to algorithmic tools which could benefit from the links. This is because algorithms operate on finite structures whereas classical Morse theory requires the setting of a differentiable manifold. An exception here is a variant of Conley theory, a substantial generalization of Morse theory. Its algebraic variant is based on homology, one of the main tools in algebraic topology. Algorithmic computation of homology, although in general not very efficient, is possible and, when combined with rigorous numerics using interval arithmetic, may provide computer assisted proofs of the asymptotic characterization of the dynamics of a differential equation. Since the beginning of the 21st century, there has been a continuous increase of important papers presenting such proofs. However, for such a proof to succeed, several parameters must be properly selected and this is achieved by many hours of numerical experiments requiring human expertise and patience. The search for an automated method of a computer assisted analysis of a dynamical system is a part of the present project. Such an automated method would have to work well with topological tools. To achieve this a combinatorial analogue of a vector field is required. Such an analogue in the form of a combinatorial vector field was proposed by Robin Forman at the very end of the 20th century, as a part of his work on a purely combinatorial version of Morse Theory. It was later generalized to the concept of combinatorial multivector field, better suited to the needs of modeling general differential equations.

The need for a combinatorial analogue of a dynamical system is even stronger visible in the analysis of sampled dynamical systems, that is dynamical systems known only from finite samples collected from observations or experiments, when no reliable mathematical model is available. This is related to our rapidly growing ability to collect huge amounts of data, both of static and dynamic nature. In geometric terms datasets form a cloud of points in a high dimensional space. The cloud approximates a closed hyper-surface in the space and the topology of this hyper-surface provides useful clues what information is hidden in the data. A very successful tool in topological data analysis is persistent homology proposed by several scientists at the turn of the 20th and 21st centuries. Persistent homology collects the birth and death times of topological features in the form of the persistence diagram or barcode. The information collected in the persistence diagram is then used to differentiate between point clouds. However, it seems that a substantially deeper insight into the geometry of the point cloud is possible. One of the outcomes of a previous project was the observation that persistent homology may be viewed as a form of combinatorial Morse Theory proposed by Forman. This observation creates a platform for the transfer of topological tools in dynamics to topological data analysis not only in the setting of dynamic data but also for general point clouds collecting static data. The information gathered this way would reveal the shape of point cloud directly, via the stable and unstable manifolds of invariant sets of the gradient dynamics and not only be based on birth-death times. To achieve this one needs to built a suitable theory originating from what is so far a mere observation. Such a theory is among the main goals of the present project.