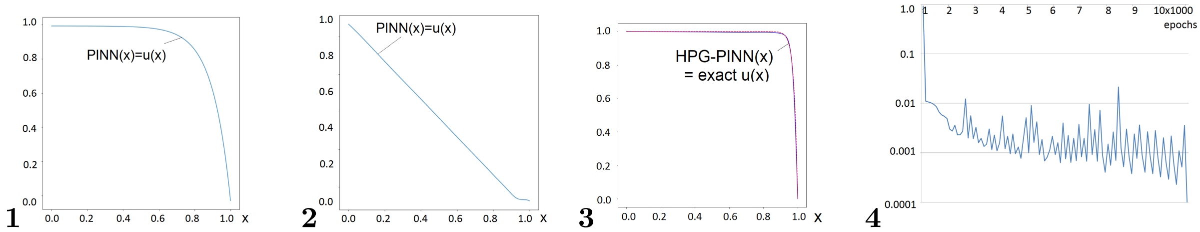

Rysunek 1: **Panel 1:** Correct solution of advection-dominated diffusion for $\epsilon = 0.1$ obtained from PINN. **Panel 2:** Incorrect solution of advection-dominated diffusion for $\epsilon = 0.01$ obtained from PINN. **Panel 3:** Correct solution of advection-dominated diffusion for $\epsilon = 0.01$ obtained from HPG-PINN. **Panel 4:** Convergence of training of HPG-PINN.

Computer simulations based on partial differential equations have a number of important social, economic and technical applications. In this project, we focus on a class of partial differential equations such as advection-dominated by diffusion equations, Navier-Stokes equations and Navier-Stokes Bussinesq equations, having applications in simulations of atmospheric phenomena. Recently, there is a large scientific interest in application of neural networks into solution of partial differential equations. A neural network can be understood as a nonlinear function $\mathcal{R}^d \ni x \to DNN(x) \in \mathcal{R}^l$ being a concatenation of linear operators and non-linear activation functions

$$DNN(x) = A_n\sigma\left(A_{n-1}\sigma\left(\cdots\sigma(A_2\sigma\left(A_1x + B_1\right) + B_2\right)\cdots B_{n-1}\right) + B_n = y \quad (1)$$

where A_i are (usually sparse) matrices of layers of neural networks, B_i are bias vectors, σ is a non-linear activation function, e.g. the sigmoid. Such a neural network can be differentiated with respect to input arguments x using automatic differentiation tools. A neural network understood in this way can learn solutions of partial differential equations, such as the advection-diffusion-dominated equation

$$u'(x) - \epsilon u''(x) = 0 \text{ for } x \in (0,1); \quad \epsilon u'(0) - u(0) = 1; \quad u(1) = 0. \quad (2)$$

This is possible by defining a loss function and using numerical libraries such as pytorch

$$LOSS(x) = \left(DNN'(x) - \epsilon DNN''(x)\right)^2 + \left(\epsilon DNN'(0) - DNN(0) - 1\right)^2 \quad (3)$$

This algorithm, called Physics Informed Neural Networks (PINN) was proposed in 2019 by Prof. George Karniadakis of Brown University. PINN methods have a number of advantages, such as affordable computational cost, ease of acceleration with the help of GPGPU cards; they are universal approximators, they allow to solve parametric differential equations "at once." However, for the aforementioned class of differential equations, the PINN method has a number of disadvantages and limitations, such as an expensive learning process due to fact that the sparsity of matrices from layers of neural networks are not taken into account; they also encounter a problem with the convergence of the learning process to a correct solution (see Figure 1).

The goal of our research project will be to develop a novel method called **HPG-PINN** (hierarchical matrices, Petrov-Galerkin-based neural network), using weak Petrov-Galerkin variational formulations, and speeding up the learning process using hierarchical matrices. The HPG-PINN method has the potential to revolutionize the way the PDEs computations are performed with neural network computation because a) by using hierarchical matrices **it will enable the fastest possible neural network evaluation** in linear cost (taking advantage of the fact that multiplication of hierarchical matrices by a vector has linear cost) which **will enable a very fast neural network training process**; (b) by using a loss function based on the weak formulation of the Petrov-Galerkin method, and calculating the optimal test functions using spectral methods, **it will ensure that the learning process converges to the correct solution for difficult problems** such as the advection-dominated diffusion problem (see Figure 1). We will use the HPG-PINN method developed in the project to simulate the process of reducing air pollution by PM2.5 and PM10 by using a shock wave generator. In our project, we plan to cooperate with Prof. George Karniadakis, the developer of PINN methods, who has expressed interest in applying the theoretical study and application of our HPG-PINN simulator to the air pollution control process. Our project will be a transfer of theoretical knowledge from the field of stabilization of computer simulations using finite element methods into the world of neural networks.