Rational homotopy groups in geometric analysis

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In physics, complex systems are often described in terms of measurable quantities that reflect some feature of the system — such as mass, charge, potential, or kinetic energy. These usually express some degree of complexity in the system, i.e., how much of something there is, or how fast it is moving or changing. In fact, the last of these quantities — energy — is central to modern formulations of mathematical physics. It is usually easy to visualize in simple mechanical systems, but less so in more subtle phenomena, such as electromagnetic or gravitational fields. Nevertheless, for example, in the case of an electric field $\mathcal{E} : \mathbb{R}^3 \to \mathbb{R}^3$ with a potential $\Phi : \mathbb{R}^3 \to \mathbb{R}$, one defines the energy of $\mathcal{E}$ over some region Ω as the averaged squared rate of change of the potential:

$$E(\Phi) = \int_\Omega |\nabla\Phi|^2. \tag{1}$$

Note that this indeed measures how "complicated" the field is: zero energy corresponds to a constant potential (and thus, no field at all), while high energy indicates a wildly varying potential and field.

For more sophisticated physical models, such as the Skyrme–Faddeev model in solid-state physics and magnetism or some theoretical descriptions of liquid crystals, one may also define other, so-called topological measures of complexity. As an example, consider a potential confined to a plane, $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$, which is restricted to take values on the unit sphere $\mathbb{S}^2$, also referred to as the target or state space. By imposing some regularity "at infinity," Φ can be viewed as a map from the compactification, resulting in $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$. For such a mapping, one can define its topological degree $\deg \Phi$, which, roughly speaking, counts how many times the domain sphere covers the target sphere — treating Φ as "transforming" one sphere into the other.

This is, again, a useful measure of complexity: a map with degree zero is akin to a constant function, while a high topological degree indicates that the potential varies significantly. It is worth noting that this invariant takes only integer values, thus exhibiting a form of "quantization."

This project investigates whether quantities measuring the topological and analytic complexity of a potential—such as degree and energy, as outlined above—are related, and if so, what the nature of that relationship is. In the present example, this is indeed the case, as the topological degree of a given mapping can be related to its energy by the following inequality:

$$|\deg \Phi| \leq \frac{1}{8\pi} \int_{\mathbb{S}^2} |\nabla\Phi|^2. \tag{2}$$

Such an estimate is possible due to a deep result from algebraic topology known as Sullivan's theorem. It connects the rational (or real) homotopy type of the target space (i.e., various generalized topological degrees) to the algebra of differential forms on that space—thus enabling an analytic estimate. More precisely, the example involving degree and energy exploits the relationship between the real homotopy type of $\pi_2(\mathbb{S}^2) \otimes \mathbb{R}$ represented by deg and the algebra of smooth differential forms on $\mathbb{S}^2$, the target space.

This construction admits far-reaching generalizations: for different target spaces, one may consider various generalized degrees arising from their real homotopy type and their relationship to different notions of energy. The appropriate notion here is the $p-$energy for different values of p:

$$E_p(\Phi) = \int_{S^p} |\nabla\Phi|^p. \tag{3}$$

A relatively straightforward procedure to obtain analytic estimates has been proposed in pioneering works by Novikov and Hardt-Rivière for simply connected manifolds. The plan for this project is to investigate for which target spaces we can extend this procedure. In particular, one can ask whether such or similar procedures can be defined for CW-complexes - a rich class of geometric spaces obtained by "gluing" together cells of different dimensions. An important example here is a wedge of manifolds - a space constructed by connecting two or more spaces at one point. Even simple examples such as $\mathbb{S}^n \vee \mathbb{S}^n$ already have new non-trivial real homotopy invariants compared to the lone $\mathbb{S}^n$.

Results of this project could lead to insights into questions in physics (e.g. existence of so-called topological solitons) and analysis (e.g. weak approximability of Sobolev maps). Additionally, relations between topological and analytical properties of maps can be used to find topological counterexamples to problems in analysis.