Algebraic geometry is a branch of mathematics focused on the study of algebraic varieties, which are sets of solutions to systems of polynomial equations. Its roots trace back to ancient Greek geometry focused on lines and conics, but it has evolved considerably over centuries to encompass much more intricate objects and advanced methodologies. The central challenge in algebraic geometry is unchangingly the classification of algebraic varieties, which is approached from two perspectives:

- Developing invariants to distinguish between varieties. This often involves abstract concepts such as derived categories a notion introduced by Grothendieck in the framework of homological algebra.
- Constructing varieties that exhibit specific invariants or notable properties, a perspective known as explicit algebraic geometry.

This project explores the interaction between classical explicit algebraic geometry and the modern framework of derived categories. By integrating classical techniques with contemporary categorical methods, the research aims to provide both theoretical insights and practical tools for understanding the geometry of varieties through their derived categories. Our specific focus is to explore the information that a variety's derived category encodes and to understand the geometric operations that induce derived equivalences.

In view of a rough classification of minimal varieties, distinguishing three classes of varieties, namely Fano type, Calabi-Yau type and general type, a central influence on the study is the Bondal-Orlov reconstruction theorem. It asserts that the derived category of any Fano variety as well as that of any variety of general type uniquely determines the variety up to isomorphism. This result distinguishes the class of Calabi-Yau type manifolds, which in the classification of varieties, lies somehow in between Fano varieties and varieties of general type. Still, unlike these other classes, two distinct Calabi-Yau type manifolds can share the same derived category, making the study of derived equivalences intrinsically tied to their theory. Functors relating two categories of different varieties are referred as Fourier Mukai transforms.

The importance of Calabi-Yau type manifolds extends beyond their role in the classification problem. These natural geometric objects arise in diverse areas of mathematics and theoretical physics, particularly in string theory, where they model the universe. String theory posits that the universe is fibered into minute, three-dimensional complex Calabi-Yau manifolds, within which strings vibrate. The vibrational patterns of strings on these manifolds determine the particles observed, making the study of Calabi-Yau manifolds crucial for understanding string-theoretic models of the universe. The interplay between the theories of Calabi-Yau manifolds and string theory has generated a wealth of hypotheses, many inspired by their physical relevance. Among these, the mirror symmetry conjecture stands out. Mathematically, it proposes that Calabi-Yau manifolds exist in pairs with interchanged structures. While some forms of this conjecture have been proven for simpler cases, such as complete intersections in toric varieties, it remains an open question for more general classes. The link between mirror symmetry and category theory has provided new interpretations and tools for addressing this conjecture. In particular, derived equivalent pairs of Calabi-Yau manifolds exhibit such striking similarities that they are predicted to share the same mirror. This observation challenges the traditional understanding of mirror symmetry as a symmetry and motivates deeper inquiry, which forms a central part of our research.

The project tackles several significant and challenging problems at the interface of geometry and derived categories:

- Investigating derived categories under birational transformations, operations which alter only a small part of the variety while leaving the rest unchanged.
- Constructing non-birational varieties with equivalent derived categories.
- Developing criteria for derived equivalence among certain Calabi-Yau type manifolds, specifically hyperkähler manifolds.
- Utilizing derived equivalences to construct hyperkähler manifolds explicitly, shedding light on these enigmatic members of the Calabi-Yau family.
- Comparing varieties through notions linked to derived categories, advancing the understanding of their relationships and classifications.

This comprehensive approach seeks to bridge classical geometry and modern categorical techniques, offering innovative pathways for both theoretical exploration and concrete applications.