1. Abstract for the general public

The present project is part of a trend of fundamental research aimed at developing mathematical knowledge in the field of Hilbert space methods and their broad applications. The research topics include issues at the intersection of the theory of operators in Hilbert spaces and the theory of C^*-algebras, drawing inspiration from both classical approximation theory and the theory of stochastic processes and quantum measurement theory. The research issues planned in the project naturally fall into three main parts, each of which leads to several key problems that remain unsolved to date.

The first research issue comes from classical approximation theory, which has its roots in the famous Weierstrass theorem on polynomial approximation of continuous functions on the interval [0,1]. In particular, it refers to Korovkin's theorem, which unifies various approximation methods, such as the polynomial Bernstein approximation or the trigonometric Fejér approximation. In 2011, Arveson was the first to transfer these ideas to the context of C^*-algebras. Based on the result of Šaškin, he provided a connection between the noncommutative analogue of Korovkin's theorem (hyperrigidity) and boundary representations (noncommutative Choquet boundary). As a result, he formulated the hyperrigidity conjecture, which has been the subject of intensive research for over a decade and remains unsolved for commutative C^*-algebras to this day. This conjecture is also related to its far-reaching generalization for commutative C^*-algebras stated by Davidson and Kennedy in 2021. The goal of our research is to verify both conjectures for carefully chosen generating sets. The expected results should also lead to a generalization of the Arveson-Brown theorem (2016), originally proved for single selfadjoint operators, to finite (or possibly infinite) systems of normal operators.

The next issue is related to the perturbation theory of linear operators, a classic issue that has been known for decades. This theory was developed by Rayleigh and Schrödinger for purposes of quantum mechanics. It allows finding an approximate solution to a differential equation (or operator eigenproblem) if the exact solution of a related but simpler differential equation (or operator eigenproblem) is known. When it comes to understanding and solving a wide range of real and natural problems, the concept of perturbation is central in mathematics, mathematical physics and related disciplines. Our research question is motivated by both some classical results and recent developments in operator theory and function theory. We are mainly interested in perturbations of order one of certain concrete operators. Our goal is to study rank-one perturbations of certain analytic operators, such as model operators, weighted Bergman shift and Dirichlet shift, among others. We expect to develop the theory along with new perturbation techniques for analytic operators.

The last of the three research topics focuses on operators with infinitely divisible formal moments. The significance of the spectral theorem in mathematics and its broad applications have served as a motivation for the search for more general classes of operators that inherit certain properties from their predecessors. Among these are subnormal operators, which are restrictions of normal operators to their invariant subspaces. According to the celebrated Embry-Lambert theorem, these operators are characterized by the fact that the sequence $\boldsymbol{\gamma}_h := \{\|T^n h\|^2\}_{n=0}^{\infty}$ of squares of norms of powers of the operator T is a Stieltjes moment sequence for every vector h. It turns out that this approach can also lead to other well-known classes of operators, including m-isometries introduced by Agler in 1990 ($\boldsymbol{\gamma}_h$ is then a polynomial in n of degree at most $m-1$). In our project, inspired by recent research on weighted shifts with infinitely divisible formal moments conducted by Benhida, Curto and Exner, we will undertake the analysis of abstract operators for which every sequence $\boldsymbol{\gamma}_h$ (h arbitrary vector) is an infinitely divisible moment sequence. We expect that one of the outcomes of our research will be the construction of integral representations for such operators.