From dense graphs to sparse ones and back again

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Already since 1970s we know that there are many natural problems, that we will (probably) never be able solve efficiently. An example of such a problem is Max Independent Set: given a graph, finds a largest set of pairwise non-adjacent vertices. The simplicity of the definition is elusive: unless something very unexpected happens, we cannot really hope for fast (or even “significantly better than brute-force”) algorithms that solve this problem, even approximately. However, such algorithms might exist if we retrict possible input instances to some family (class) of graphs.

The study of computational problems restricted to specific graph classes has been an active research direction in past few decades. Quite often in this, and also many different contexts, we witness a certain duality between sparse graphs and dense graphs. There are many reasonable definitions of “sparse,” but in general[1] classes of sparse graphs are considered to be well-understood. They often admit some strong structural properties which allow us to, e.g., bound the chromatic number. In many cases these properties can also be exploited algorithmically. In particular, many classic computational problems, hard for general graphs, become much easier for sparse graphs classes (if not solvable efficiently, then at least admitting good approximation algorithms). In contrast, dense graphs are considered structurally complicated. Furthermore, from the algorithmic perspective, they are often a good place to look for intractability results.

However, it turns out that even among classes of dense graphs we can find many examples that are well-structured. Intuitively speaking, we will be interested in classes of graphs that **“only pretend to be dense.”** More formally, there are classes of (possibly dense) graphs that admit some sparse underlying structure. We want to investigate how the results developed for sparse graphs classes can be generalized for such “structurally sparse” classes. We are in particular interested in transferring the tools from one world to another.

Moreover, we are also going to look at classes of graphs that **“only pretend to be sparse.”** More specifically, we are going to investigate (possibly sparse) graphs with a total order on vertices. The total order relation is dense, but still not very complicated. Due to the first property, even very simple classes of graphs (say, matchings) have quite rich structure in the ordered world. Due to the second property, we might still hope for some interesting structural and algorithmic results.

[1] greatly oversimplifying