Substructures in combinatorial objects: the deterministic and the random setting (description for the general public)

The ultimate objective of this grant proposal is to make further progress in finding and measuring some "better organized" substructures of combinatorial objects such as permutations, words over finite alphabets, graphs and hypergraphs. We propose four research tasks:

#1 Substructures in (random) permutations, ordered matching, and words over a finite alphabet,

#2 Properties of randomly augmented graphs and hypergraphs,

#3 Rainbow structures in dense colored hypergraphs,

#4 (Semi)-random hypergraph processes.

All tasks belong to the mainstream of contemporary combinatorics. They have a long history reaching back to a century ago and beginning with Ramsey and Turán theorems for graphs and hypergraphs, Erdős-Szekeres theorem about monotone subsequences and, much earlier, the theorem of Thue constructing square-free ternary words. Several other biggest names in mathematics have been associated with some of these problems, most significantly, Alfred Rényi, Béla Bollobás, Endre Szemerédi, and Noga Alon.

Under research task #1 we will focus on homogeneous subsequences and twins (i.e. pairs of disjoint isomorphic sequences) in permutations, ordered matchings, and words over a finite alphabet. Planned results on homogeneous subsequences are analogs of the celebrated Erdős-Szekeres Theorem. In turn, looking for twin objects in mathematical structures has long and rich tradition, going back to some geometric dissection problems that culminated in the famous Banach-Tarski Paradox. A general problem is to split a given structure (or a pair of structures) into as few as possible (pairwise) isomorphic substructures. Here we study a related question: how large disjoint isomorphic substructures can be found in a given structure?

The second research task is devoted to the study of randomly augmented hypergraphs which is a relatively new but quite "hot" topic interpolating between random graphs and extremal problems, two major branches of modern graph theory. A mixture of a deterministic and a random structure seems to be a fascinating object to study with a potential of applications. We would like to determine the whole spectrum of behavior of graphs under this model for two graph properties: the appearance of high powers of Hamilton cycles and clique factors.

In research task #3, rainbow analogues of Turán and Dirac problems can be viewed as a measure of robustness of certain graph properties under properly bounded edge-colorings. This branch of graph theory has been extensively studied in recent years. Our aim is to determine rainbow Turán numbers for certain classes of graphs such as paths and cycles. We would also like to give more evidence for the folklore conjecture stating that Dirac thresholds are asymptotically preserved in the rainbow setting.

Finally, under the fourth research task we intend to study two random hypergraph processes with restrictions, which constitute more sophisticated and, at the same time, more "real life" oriented models of an evolving random graph. The first of them puts a cap on the maximum vertex degree and its study originated in a question of Erdős from the eighties about the probability of saturation of all vertex degrees. The other one, only recently developed semi-random process, allows one to build a graph or hypergraph, alternating random and deterministic steps, and thus falls into the rich and popular area of games on graphs.