Decision Problems for Logics with Weak Forms of Recursion
Description for the general public

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The main problem. The topic of the Preludium Grant lies within theoretical computer science and can be placed in the intersection of mathematical logic with practical fields such as knowledge representation, database theory, or formal verification of software. We aim to study the *satisfiability problem*, frequently referred to as *the classical decision problem*. The satisfiability problem is parametrised by some formal language (a.k.a. *logic*) $\mathcal{L}$ and asks whether an input formula φ (written in the syntax of $\mathcal{L}$) holds true in *some* structure (*i.e.* it has a *model*).

(Finite) Satisfiability Problem for $\mathcal{L}$

Parameters:	A logic $\mathcal{L}$.
Input:	An $\mathcal{L}$-formula φ.
Question:	Does φ have a (finite) model? Alternatively, is φ satisfied in some (finite) structure?

We also consider the *finite* satisfiability problem, where the desired model is supposed to be of finite size. If every satisfiable $\mathcal{L}$-formula has a finite model, we say that the logic $\mathcal{L}$ has the *finite model property* (FMP).

The satisfiability problem find many applications. First, it was used as a yardstick measuring the "quality of logic" in a quest for identifying important fragments of first-order logic. This has been a central undertaking of mathematical logic since the appearance of Hilbert and Ackermann's *Grundzüge der theoretischen Logik* almost a century ago. For such fragments (query languages), the satisfiability problem can be used as a fundamental algorithm for query optimisation (*e.g.* the problem of whether two queries are equivalent reduces to satisfiability). Second, in the area of *formal verification* of software and embedded systems, the satisfiability problem can be used for checking the presence of unwanted behaviours of the system, *e.g.* if it enters a dangerous zone or if the system does not contain deadlocks. Finally, the satisfiability problem is often employed in *logic-based knowledge representation*. Therein we axiomatize and standardize domain knowledge by means of *formal ontologies*, which are just collection of formulae, usually written in *description logics* (DLs) — a robust family of logical formalisms serving as the logical foundation of contemporary standardised ontology languages, including OWL 2 Web Ontology Language by the W3C. The satisfiability problem for DLs can be employed to automatically verify the consistency of the collected knowledge as well as for knowledge extraction and querying.

Our choice of $\mathcal{L}$. Among many features offered by extensions of the basic description logic $\mathcal{ALC}$ (from the family of logics discussed above), an especially useful one is $\cdot_{\mathsf{reg}}$, supported by the popular $\mathcal{Z}$-family of description logics. With $\cdot_{\mathsf{reg}}$ one can specify regular path constraints by means of regular expressions with tests, allowing the user to navigate through graph-structured data. The $\cdot_{\mathsf{reg}}$ feature can be also seen as *weak form of recursion*. In recent years many extensions of $\mathcal{ALC}_{\mathsf{reg}}$ (the core fragment of the $\mathcal{Z}$ family) for ontology engineering and formal verification were proposed, and practical algorithms for reasoning in (the extensions of) $\mathcal{ALC}_{\mathsf{reg}}$ are known. The logics from the $\mathcal{Z}$ family of DLs are among the most powerful knowledge representation formalisms on the verge of decidability, and encode (suitable fragments) of the Web Ontology Languages by the W3C. What is more, $\mathcal{ALC}_{\mathsf{reg}}$, turns out to be a notational variant of Propositional Dynamic Logic – a popular and well-studied program specification language. This makes the $\mathcal{Z}$ family of DLs a desirable object to study.

Research questions. We plan to investigate the satisfiability problem for various extensions of $\mathcal{ALC}_{\mathsf{reg}}$ are still open. Such problems are both practically relevant (*e.g.* for the area of formal verification or knowledge representation), while at the same time they are mathematically challenging. Our aim is to (i) obtain coNP-completeness (w.r.t. the data complexity) for the entailment of positive two-way regular path queries over tamed $\mathcal{ZOIQ}$, (ii) find a decidable higher-arity extension of $\mathcal{ALC}_{\mathsf{reg}}$ that has the FMP, and (iii) solve the long-standing open problem of whether the finite satisfiability for $\mathcal{ALC}_{\mathsf{reg}}$ with the "Strong Loop Predicate" is decidable.