Large configurations of rational curves on certain algebraic surfaces

description for general public

Objectives of the project

This project is devoted to algebraic geometry. We want to determine the maximal number of conics and smooth rational curves of a fixed degree on $K3$-surfaces of high degree. We intend to examine surfaces with singular points and surfaces defined over fields of positive characteristic as well. Furthermore, we intend to find maximal number of twisted cubics (the most well-know rational curves apart from lines and conics) on a smooth complex $K3$-surface of degree six. The methods we apply should allow us to find all $K3$-sextics with maximal number of twisted cubics.

Moreover, we want to prove a bound on the number of lines on smooth surfaces of degree at least five in three-dimensional complex projective space. Such surfaces are given by one equation, which simplifies certain reasonings. They are no longer $K3$-surfaces, so they require entirely different techniques.

Reasons for choosing the topic

It is known that no formulae for roots of polynomial equations of degree at least five can be found, since the XIXth century. Mathematicians circumvent this problem by describing the properties of sets of solutions of systems of algebraic equations. In particular, in dimensions one, two, three algebraic geometers managed to divide such sets of solutions into various classes of solution sets with similar properties. One of such classes are $K3$-surfaces.

$K3$-surfaces can be seen as two-dimensional analogues of elliptic curves (geometric objects that resemble donuts). They are also of interest for theoretical phycisists (as Calabi-Yau varieties of small dimension). Within the last decade the geometry of $K3$-surfaces with many lines has been well understood. Although some of the results were obtained by computer-aided symbolic computations, most interesting facts were derived by human mind using algebraic geometry and algebra. Smooth rational curves of degree d are natural generalisations of lines (lines are degree-1 rational curves). Within the scope of the research project we intend to study rational curves of various degrees on $K3$-surfaces, with special emphasis on conics and twisted cubics ($d = 2, 3$).

The curves and surfaces to be examined in the framework of the project were studied for the last 150 years and our purpose is to fill some gaps in the classical theory, which still undergoes intensive developement. One should point out that surfaces with many rational curves are also applied to construct various mathematical objects with unexpected properties. Therefore, profound understanding of the geometry of that class of surfaces is very valuable for testing conjectures and seeking unexpected relations between various mathematical objects.

Research to be carried out

We intend to apply various techniques of algebraic geometry and algebra. One of the ideas is replacing the $K3$-surface in question by a certain vector space of dimension 24, so that rational curves correspond to certain vectors of given length. Another idea is change of the base field - i.e. replacing the set of considered numbers by a smaller one without changing abstract properties of the set of solutions. Within the scope of the project we will also carry out symbolic computer-aided computations.