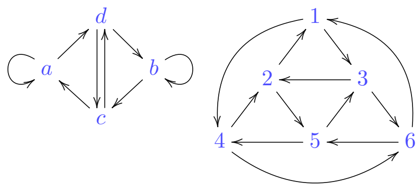
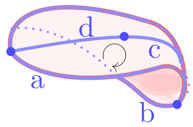
Project is devoted to topics at the crossroads of algebra and representation theory, which are entwined with combinatorisc of triangulated surfaces in a very interesting way (by a triangulated surface we mean a 2-dimensional compact manifold with fixed triangulation). Notion of representation (originally, for groups) was first studied by Frobenius in the second half of XIX century, and he probably didin't even expect that several decades later it would find applications in quantum mechanics (description of orbitals), or in the theory of molecule vibrations. Representations (later also called modules) played an important role in the developement of mathematics – especially algebra – and from early 70's of XX century they constitute separate object of study in so called representation theory of algebras. We consent this date of birth, since we owe its fruitful growth from 70's to the introduction of notion of quiver, which revolutionized approach to desrciption of algebras and their module categories, enriching it in a new combinatorial way.

Quivers are just directed graphs, but with loops and multiple arrows allowed. By a description of an algebra A (over a field K) by a *quiver with relations* we mean to give the quiver Q and certain collection of equalities called *relations*, in such a way that given algebra is a quotient $A \cong KQ/I$, where KQ is so called *path algebra of* Q (its basis is given by all paths in Q), and I is a twosided ideal in KQ generated by relations (which are linaer combinations of paths, such that the generator $\rho \in I$ corresponds to the equality $\rho = 0$ in A).

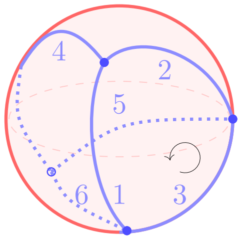
Examples of quivers



Triangulation of disc



Triangulation of shpere



Our main object of study are algebras associated to triangulated surfaces, also knows as *weighted surface algebras*, whose definition was a result of synthesis of many years of research on so called *symmetric* and *periodic* algebras. The latter have been attracting attention of mathematicians since at least late 30's of XX century. In the construction of a weighted surface algebra we start with a triangulated surface and we pick arbitary orientation of any triangle in triangulation. Then we associate to it a *triangulated quiver* Q, whose vertices are edges of the triangulation and arrows in Q reflect chosen orientation of triangles. Then the weighted surface algebra (for this surface) is defined as a quotient KQ/I, where I is generated by 'relations' depending on mutual position of triangles in triangulation and some weights. For example, picture at the top shows triangulated quivers assosiated with triangulated surfaces from the picture on the left (we choose so called *coherent* orientation of all triangles, that is every common edge of two triangles has opposite orientation in both triangles).

Recent results obtained in 2019 show universal character of the above construction. Namely, it has been proven that, if A is an algebra of the form $A = KQ/I$, where Q has at least 4 verices and it is 2-regular (i.e. every vertex in Q is a source and target of of exactly two arrows), then A is a tame symmetric algebra of period four if and only if A is a weighted surface algebra (besides few exceptions) or it is so called *higher tetrahedral algebra* (the latter is given by the second quiver at the above picture, but the relations are different from the case of surface algebra defined on this quiver, and the resulting algebras are non-isomorphic). Moreover, in 2021-2023 we have been exetensively studying possible extensions of these results, and in consequence, we have discovered two new classes of algebras essentially generalizing weighted surface algebras. The widest know class was called *generalized weighted surface algebras*, because their construction is a subtle counterpart of the previous construction, in which we begin with triangulated surface.

The aim of this project is to present full classification of all tame symmetric algebras of period four, which constitute important object of study related to various areas of algebra. Such a classification would be a solution of a long-standing open problem, and simultanouesly, a capstone of many years of research. On the other hand, recent results show very clearly that the class of generalized weighted surface algebras is a natural candidate. Main conjecture of this project is that every tame symmetric algebra of period four (except a few minor cases) is a generalized weighted surface algebra. Besides proving this, project is focused on studying the above mentioned classes of algebras from an angle of a modern notion of *mutation*, which has natural interpretation both in the language of flips of triangulations and in the language of homological algebra.