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PDEs in complex and convex analysis and geometry

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Abstract for the General Public

For an average person complex numbers are very abstract, or even *imaginary*. However, they play a central role in mathematics and are often used in physics and many other sciences (although sometimes researchers in those fields are probably not even aware of their significance). Many mathematical problems, often formulated in an elementary way and understandable to such an *average person*, e.g. in number theory, algebraic geometry or differential equations, are solved with a crucial use of complex numbers. It is no coincidence that for several hundred years the leading mathematicians like Euler, Gauss or Riemann, among other things, have crucially contributed to complex analysis. An anecdote attributed to the famous (at least in Japan) mathematician Kiyoshi Oka says that during a class with his students he would draw on a blackboard a disk that would symbolize the whole mathematics and explain that its interior is complex analysis whereas the boundary corresponds to the rest of mathematics.

This project deals with two main partial differential equations in the analysis of several complex variables: the $\bar\partial$-equation (also called the inhomogeneous Cauchy-Riemann equation) and the complex Monge-Ampère equation. The theory of $\bar\partial$-equation and related L^2-estimates, initiated by Hörmander in the 60s, allow in particular to construct holomorphic functions in a nontrivial way. One of potential applications outside of complex analysis could be an attempt to prove the Mahler conjecture from convex analysis, formulated in the 1930s. It says that for a symmetric convex body in $\mathbb{R}^n$ the minimum of the product of its volume with the volume of its dual is attained for the n-dimensional cube. So far it has only been proven in dimensions 2 and 3. About 10 years ago Nazarov obtained an equivalent formulation of this problem, now the question is to construct a holomorphic function in $\mathbb{C}^n$ with appriopriate L^2-estimates. It seems that that Hörmander's theory for $\bar\partial$ together with recent results in this area give a good opportunity to investigate this equivalent Nazarov formulation and related problems.

The complex Monge-Ampére equation is a fully nonlinear elliptic partial differential equation of second order. The nonlinearity makes it much harder to solve, the solutions to the Dirichlet problem cannot for example be given by a representation formula (as is the case with the Poisson equation which is linear). Existence of such solutions is proven indirectly, using functional analysis and the so-called continuity method, the problem is reduced to a priori estimates for solutions. This way Yau in 1970s proved the Calabi conjecture which asked for existance of Kähler metrics with certain curvature properties. Thanks to this the Monge-Ampère equation plays a central role in complex geometry. It is also crucial in pluripotential theory (that is potential theory in several complex variables), in this case we are looking for weak, that is not necessarily smooth, solutions.

One of the aim of this project is to investigate regularity of solutions as well as possible complex isoperimetric inequalities related to this equation. They would be a counterpart of the classical isoperimetric inequality (which gives an optimal upper bound for the area of a domain on the plane in terms of its perimeter) as well as Aleksandrov-Fenchel inequalities for convex bodies in $\mathbb{R}^n$. Such a complex isoperimetric inequality would in particular imply important estimates for the complex Monge-Ampère operator and would give new solutions of this equation.