HOMOTOPY THEORY IN ARITHMETIC GEOMETRY VIA CONDENSED MATHEMATICS

DESCRIPTION FOR THE GENERAL PUBLIC

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In this project, we plan to study schemes, varieties and rigid spaces – basic objects of algebraic and rigid geometry, via novel methods combining *condensed mathematics* and *higher homotopy theory.*

Varieties can classically be thought of as zeros of systems of polynomials. Over complex numbers, such objects give rise to meaningful topological spaces (i.e. certain concretely defined "shapes" that mathematicians can study) and the tools of algebraic topology are available to examine this "shape" at hand.

However, the algebraic structure that varieties come equipped with not only gives us extra tools for this study, but allows us to carry on similar research in situations where, at first sight, there is little to no topological information: for example, over finite fields, i.e. in situations where the field of complex numbers $\mathbb{C}$ is replaced by a field that has only finitely many numbers in it – something of a much more arithmetic nature.

This was possible thanks to ideas going back to the pioneer of the field – Alexander Grothendieck, who envisioned creating "shape" out of purely algebraic data and mimicking algebraic topology techniques on it with a great success. The underlying "artificial topology" was called the *étale* topology. This allowed to recreate notions known from algebraic topology such as the *fundamental group* (which intuitively parametrizes "loops" inside of our space) in the form of the *étale* fundamental group $\pi_1^{\text{ét}}(X)$ of a variety X. Later developments by Artin–Mazur–Friedlander allowed to do the same for the *higher* homotopy groups: $\pi_2, \pi_3, \ldots$.

In a series o works, Peter Scholze, together with collaborators, realized that working in the seemingly more complicated *pro*-étale topology actually simplifies many parts of the theory. Moreover, these ideas led to the nascence of the so-called *condensed mathematics.* The basic premise of this theory is that by considering mappings from seemingly esoteric (although relatively "basic") topological objects – *profinite sets* – to topological spaces, largely allows one to work with them as if they were just sets and, if they moreover carry an additional algebraic structure, as if they were purely algebraic objects.

FIGURE 1. Converging sequence of points together with its limit – a basic example of a profinite set.

Our goals in this project can be briefly described as follows:

- **Use condensed mathematics to define and study finer invariants:** for example, use it to define higher homotopy groups (as Artin and Mazur did) but in the *pro-étale* setting. Compare them to the more established notions. Compute these invariants in some cases.
- **Use condensed mathematics to bring together ideas from various fields of mathematics**: try to explain phenomena from algebraic/rigid geometry and topology within a joint and general formalism.
- **Study *how large* these invariants are and how much information they carry:** it seems that the condensed π_1 can get quite large without passing to a certain quotient! On the other hand, we plan to show that certain "tame" versions of the π_1 are more reasonably sized than previously expected.