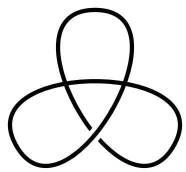
EXISTENCE AND REGULARITY OF FRACTIONAL HARMONIC MAPS

KATARZYNA EWA MAZOWIECKA

According to legend, in the year 333 BCE, Alexander the Great reached Gordion, where the famous Gordian Knot was located. According to prophecy, whoever could unravel the knot would become the king of all Asia. While in the midst of conquering Asia and seeking to strengthen his position, Alexander decided to solve the problem. Realizing that the knot could not be untangled conventionally, he chose to cut it instead.

From a mathematical perspective the problem of unraveling the knot is interesting. Mathematically, Alexander might have been correct — there exist knots that cannot be untangled. One example is the trefoil knot,



which cannot be continuously mapped into a circle. This implies that the trefoil knot and the circle are not homotopic. In this project, i.a., we will explore fractional harmonic maps related to Möbius Energy (a functional defined on the space of knots), used, e.g., in mathematical models of DNA. We will consider the minimization problem among maps that are homotopic to a fixed map.

The fractional harmonic energy can be seen in the following context. Imagine we have a wire and we glue both ends of the wire to a surface (for example to a sphere) and then we dip it into a soapy water. How does the soap film touch the surface? It turns out that the mathematical equation describing this boundary is the equation of a half-harmonic map (a critical point of the half-harmonic energy).

In this project, we will be interested in maps between Riemannian manifolds Σ and $\mathcal{N}$. Manifolds are geometric objects, such as a smooth surface without self-intersections, like the surface of a ball, ellipsoid, or a bicycle inner tube. Given a smooth map $v\colon \Sigma \to \mathcal{N}$, we wonder if there exists another map that minimizes a specified energy among maps that are homotopic to v. The concept of homotopy needs to be understood appropriately here. The maps we consider are not necessarily continuous; they are only in the Sobolev space. Homotopy theory can be extended to an appropriate class of such maps — those belonging to the critical Sobolev space.

We will explore several energies mathematically related to each other: harmonic energy, i.e., the Dirichlet integral, its generalization, the p-harmonic energy

$$E_p(u) := \int_\Sigma |\nabla u|^p dV,$$

and the fractional harmonic energy (where the considered energy is a Gagliardo semi-norm). These are fundamental functionals considered in calculus of variations. The main difficulty in studying these objects lies in the nonlinear constraint — the map u takes values in the manifold $\mathcal{N}$, making the problem nonlinear, and classical regularity theory does not directly apply. In particular, maps we consider may be singular (i.e., not continuous). One of the project's goals is to create tools that will be applied in knot theory and surface energy theory as well.