Complexity and describability in Analysis: from freeness to polishness

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The so-called Banach–Tarski paradox is a highly surprising theorem, stating that a ball in three-dimensional space can be divided into five pieces, which, with appropriate rotations and translations, can be reassembled into two identical balls to the original. The apparent paradoxical nature of this theorem lies in the fact that these pieces are, in a strong sense, *non-constructible* (one cannot assign them a three-dimensional volume); hence, for instance, one cannot enact their construction in physical reality. Constructibility (even in a more restrictive sense) lies at the heart of a branch of mathematics called descriptive set theory, which seeks to study the complexity of subsets of Polish spaces, i.e., mathematical objects that, figuratively speaking, are no more complicated than the set of real numbers. Investigating the complexity of problems concerning, for instance, the classification of mathematical objects, allows for the distinction of those that are elusive due to their inherent non-constructibility from those which also possess certain additional desirable properties, associated with concepts such as computability.

Mathematical objects of a specific type can be themselves points of Polish spaces, allowing consideration of issues related to the complexity of these properties. In Functional Analysis — a branch of mathematics dealing with linear and continuous transformations of spaces (usually of infinite dimension) — this idea has been applied to three fundamental classes of objects: separable Banach spaces (the most fundamental objects from a functional analysis standpoint), separable C*-algebras and von Neumann algebras that can be represented on separable Hilbert spaces (these last two classes of objects were introduced to formalise quantum mechanics). For C*-algebras, there are several approaches in the literature to constructing Polish spaces composed of these, but to this day, it is unknown whether the complexities of given problems described in different Polish spaces depend on them or not. All constructions mentioned here boil down to noticing that certain objects can be embedded in a larger (universal) object, naturally laying the foundation for further construction (for separable Banach spaces, this possibility stems from the theorem proven by Godefroy and Saint-Raymond in 2018). However, this approach relies on such classes of objects, which do not have a natural universal object in this sense, in which (separable) Banach algebras (Banach spaces with a specific multiplication that is continuous) embed.

The primary objective of the project is to develop a unique approach to constructing such Polish spaces made up of mathematical objects, which will be based on precisely the opposite approach. Instead of expecting the existence of an object that is universal with respect to embeddings, we will consider sufficiently large so-called free objects for these classes of objects, characterised by the fact that they can be projected onto all objects in the considered class. A central hypothesis regarding this approach, aside from a certain unifying value for subfields of functional analysis, is the potential simplicity in achieving new results reducible to describability or constructibility of sets of objects having specific properties, which may possibly result in a more efficient classification (or demonstrate its impossibility) of various mathematical structures appearing in mathematical analysis, but also in other areas of Mathematics.