Summary

The past decades of research on the mathematical foundations of fluid mechanics have focused on two fundamental systems of partial differential equations - the Euler system and the Navier-Stokes system. These equations attempt to describe the phenomena of fluid flow in a simplified way. In the case of flows in geometrically simple domain or fluids with rather idealized properties, this program actually has a chance of great success. However, when faced with real-world situations, this description is simply not enough. The unsatisfactory physical description is followed by the negative mathematical results that have so recently taken the scientific community by surprise. Initially, the innovative methods of convex integration indicated the ill-posedness of the incompressible Euler equations - they showed the existence of infinitely many solutions with low regularity, having the property of producing energy. Undoubtedly, this phenomenon can be called a mathematical perpetual motion machine. In the following years, these methods were developed for the Navier-Stokes system.

The most important role of models of physical phenomena is to predict reality. Verification of this key role is not always easy. Mathematical analysis of solutions can help here, which can indicate, as in the case described above, some pathological behaviour of solutions. It finds unexpected properties of solutions that would never be observed in a real-world physical situations. Such mathematical results are a red light for the model - a signal that, at least for a certain range of parameters, the solutions may be far from the reality they are supposed to describe. Notably, in the case of nonlinear models, the risk of such situations is significantly higher.

Therefore, the research planned in this project will go beyond the traditional framework. We will identify systems of equations of significant complexity. We will take into account the influence of such factors as the electric field, capillary effects, the effect of fluid vorticity, collective behavior of individuals, phase transitions, or the important internal structure of the fluid. The methods that will be developed are currently highly appreciated in the fluid mechanics research community. Due to the complex nature of the equations, nontrivial nonlinearities in systems of equations are a common challenge. They can make the standard compact methods insufficient for the limit passages. One of the most fundamental questions in mathematical physics and biology is the problem of relating different models, usually describing the same phenomenon but at different scales. This question originates from Hilbert's sixth problem, namely, the rigorous passage from the Boltzmann equation describing the matter at the level of particles to solutions of the Euler equations describing compressible gas dynamics at the macroscopic level. In recent years, Hilbert's problem has been viewed in a broader way. It has been linked to several different equations, motivating new groundbreaking steps in nonlinear analysis.

Thereby one of the key goals of this project is to find rigorous relationships between different models and to observe the dependencies between their solutions. In particular, noticing that models of biological processes are ideologically very close to models of fluid mechanics.