Rigorous numerical methods for Functional Differential Equations

Ordinary Differential Equations (ODEs) are among the basic models for precise and systematic investigation of real-life phenomena. To get an idea what those models look like, let imagine driving a car. Using steering wheel, breaks and accelerator we can change the speed and the direction in which we are moving - in other words, we give the car a velocity vector (v). Of course, in deciding what to do next, we take into account the current state at the position we are in (x), therefore the velocity is a function of our position $v = v(x)$. Now, imagine that the car is leaving a single point mark on the road, so the line drawn by it (*the trajectory*) depends on the trip time: $x = x(t)$, and it has the property that the current velocity vector is tangent to it at each point during the move. In other words, the velocity is the derivative of position with respect to time: $x'(t) = v(x(t))$. Thus, x is a solution to an equation involving position and its derivative - an Ordinary Differential Equation.

Driving a car we try, based on the current state, to optimise the velocity to safely reach our destination. But do we make decisions outright? Obviously, the answer is „no", among others, due to the construction of our neuronal system, we always react with *some delay*. One can easily see that this happens in other situations in our life too: the birthrate of new organisms usually does not depend on the current population but on the population of mature specimens, which need time to grow up. Gene regulatory mechanism in our body take time as the substances need to be transported to right cells. The speed of electromagnetic waves is finite, so sending commands from Earth to a robot on Mars has at least three minutes delay. Therefore, to investigate realistic models of real-life phenomena we need to include some form of delay.

There are many forms of the delay to consider: constant delays, state dependent delays (e.g. a signal needs more time to travel greater distances), or the distributed delays (organisms can mature after a different period of time). A common term to equations with different forms of delay is Functional Differential Equations (FDE). On this class of models we want to focus our attention in this project.

There are many important questions one may ask in models with delay. How does the delay influence the model? Is it possible to predict the trajectories of the model? Is it stable or do the solutions seems to be chaotic? Does a small change in parameters can lead to a catastrophe? How big delay can we set before the system becomes unpredictable (e.g. if some drug increases our reaction time, is it safe to take it and drive)? Answering such question is crucial to understand the long term behaviour of a model (its dynamics). And the dynamics of the model often allows to give an explanation of the real-world phenomenon and predict its future state.

Unfortunately, most of the Delay Differential Equations (DDEs) modelling the physical phenomena cannot be tackled directly by the rigorous mathematical reasoning. For such systems the computers can be used to study their behaviour numerically, by using specially designed simulation software. However, numerical solutions are always only an approximation of the true dynamics. So here comes the question: could we somehow verify if the picture obtained by computers are close to the true solutions? The answer is positive in many cases and one of the possible tools to achieve this are the rigorous numerical methods, which aside finding the approximate solutions also prove that sufficiently close to this approximation there is a real one, and the two looks almost the same. This kind of methods were successfully implemented in the case of ODEs and currently there is a lot of effort in extending them to other, more difficult classes of problems.

The Aim of this project is to design and implement them in the context of FDEs by providing necessary mathematical tools and computer software for rigorous, computer assisted verification of numerical simulations. Up till now, authors of the project successfully designed such methods for the simplest case of constant delays and our research indicate that the extension to a more general class of FDEs is possible. We hope that the results obtained in this project will not only be usable to prove important mathematical theorems, but they will become in the future an useful tool for the researchers in other branches of science.