

”Functional calculi for semigroup generators and related matters”
(popular description)

The research proposal is in the area of pure mathematics. It mostly concerns the theory of linear mappings on infinite-dimensional spaces. This is a kind of abstraction helping to provide a unifying look at many seemingly distant issues in mathematics and thus leading to their better understanding. The project aims at uncovering subtle properties of the iterates or more generally, semigroups of those maps and at relating the properties to another abstract concepts of operator theory such as functions of operators, a general view on the notion of function. An important part of the project is to find applications of abstract results to the problems arising in natural sciences and described by partial differential equations. The value of the project is ”twofold”. The sense of the project ”per se” is that it will lead to essential internal developments in mathematics and might be a base for the progress of mathematics as a science. The practical applications of mathematical results are not immediate and could await for their real world use for decades if not centuries. However, while the precise description of the place and merit of the project for wide community would take us far away and it is hardly possible in a short text, in the long run, the project will help eventually to provide insights on various phenomena of natural sciences related to the real world. Rather than to give a full account we provide several instructive illustrations. The evolution of many systems in physics, biology, chemistry, engineering or economical sciences can be rewritten in the mathematical language of partial differential equations or - more abstractly - of evolution equations on Banach spaces called abstract Cauchy problems. One could think of diffusion processes, transport processes, wave propagation phenomena, the dynamics of electromagnetic waves, the motion of elastic or viscoelastic materials, phase separation processes, population dynamics and epidemic models. The mathematical analysis of these models allows us to understand their qualitative behavior such as regularity of solutions, positivity, or asymptotic behavior. The primary interest in the study of strongly continuous operator semigroups, one of the basic object of operator theory, comes from the fact that such semigroups solve abstract Cauchy problems which are often models for various phenomena arising in natural sciences, engineering and economics. The semigroup orbits are precisely so called mild solutions of the corresponding abstract Cauchy problems. Knowledge of the orbit properties and their nature allows one to characterize short-time and long-time evolution of these phenomena, thus linking abstract considerations to their concrete applications. Roughly, operator semigroups can be viewed as exponential functions of their generators, thus generalizing the well-known notion of exponentials. This way of looking at the semigroups brings us to the area of analysis called functional calculi. That area is intimately related to semigroup theory, and provides tools, methods and objects helping to express the properties of various systems in adequate and exhaustive way by means of semigroup generators, their functions and related objects. It will be a crucial part of our project. As exponential is a kind of universal and omnipresent object in mathematics, one should not wonder that operator semigroups share this feature too. Our project will depend on the input of a variety of mathematical disciplines including harmonic analysis, complex analysis, partial differential equations, to mention a few. It is important to note that the semigroup theory provides a rigorous mathematical base for describing natural phenomena, and it is often a foundational building block for modern science. Think of quantum systems as an example. The evolution of quantum systems are usually recasted in the realm of operator theory, and physicists often rely their intuition on such an abstract approach. While our project has no immediate links to this kind of mathematics it adds various bites to techniques and insights used by natural scientists. There is a variety of applications of semigroup theory ranging from engineering, that includes control theory, signal processing, networks dynamics, numerical analysis and modeling, to the studies of dynamics of population, cancer deceases, evolution of cells in mathematical biology. Those applications are sometimes quite unexpected: it suffices to remind that the solution of the famous Kato ”square root” problem appeared to be very useful ... in computer tomography. The latter shows that it is quite hard to predict the future of mathematical results, and their real merit uncovers sometimes in unpredictable ways. What can really be definite is the contribution of the project to mathematics as a part of human culture and a number of areas within it.