Post completeness vs neighborhood completeness in classical modal logics

Commonsense understanding of a term ‘logic’ states that logic is a science concerned with reasoning, especially with deduction. In a more narrow sense, logic is a set of sentences closed under some rules. This technical approach to logic is somehow related to the basic intuition: logic is a system which is supposed to formalize correct reasoning from everyday life. Classical logic, despite its many merits, fails to capture all of such reasonings. This fact justifies the need for the so-called non-classical logics. One of such logics is modal logic, which enriches the basic language of classical logic with modal operators. These operators correspond to terms of natural language which can be expressed with the words “it is necessary that…” and “it is possible that…”.

Modal logic itself is in fact a gigantic family consisting of infinitely many logics. Among them, we distinguish normal modal logics whose metatheory, after several decades of investigations, reached the status of independent scientific field. It turns out however, that the set of normal modal logics is nothing but a small fragment of a much larger family of the so-called classical modal logics. Classical modal logics constitute the widest known family of modal logics. In this sense, investigations of this family lead to the most general theory of modal logic per se. Nonetheless, one will not find many publications on classical modal logic in the literature. Thus, there are unanswered questions to be found and many problems remain open.

The aim of the project is to answer two questions which seem to be tightly connected to each other. The questions were posed by Peter Fritz in 2016. The author asks to give the exact number of classical modal logics which are Post complete and neighborhood complete, and the exact number of such classical modal logics which are Post complete, but do not have any class of adequate neighborhood frames. In order to familiarize the reader with the significance of both problems, let me try to explain two key notions.

Firstly, Post completeness can be understood as a certain type of maximality. Namely, a given logic is Post complete when it cannot be properly extended without falling into inconsistency. In other words, a logic which is Post complete has only one proper extension: the inconsistent logic. Post complete logics play a crucial role, because having so many theorems, they provide the richest tool of reasoning.

Secondly, neighborhood structures are mathematical objects which define the truth conditions for formulas of modal logic. It is possible that there exist logics which do not have an adequate class of neighborhood structures. This would mean that there are logics for which we cannot find a class of structures which would satisfy all and only those sentences which are its theorems.

The potential answers to questions formulated by Fritz would shed light on the possibilities and limitations of neighborhood semantics with respect to classical modal logics which are maximal and consistent.