Computable structure theory, and philosophy of mathematical structuralism

What is mathematics about? The answer that comes to mind is that mathematics is about numbers. However, we stop being so sure once we think of graph theory or group theory. In the philosophy of mathematics, a view has emerged, dating back to the 1960s, that mathematics is about structures (in other words - models), such as natural numbers, graphs, groups, etc. At the core of structuralism is the belief that mathematicians do not distinguish between systems that have the same structure (or, in mathematical jargon—which are isomorphic). What matters to mathematicians is not the inner nature or representation of the objects they study (a question like, "What is the number 5?" is not scientifically relevant), but only the relations that hold between them. This philosophical view is known as mathematical structuralism.

This scientific project addresses four selected concerns related to mathematical structuralism and proposes to resolve them using tools of a mathematical theory known as computable structure theory.

One of the popular views in mathematical structuralism is the so-called computational structuralism. It claims that intended models of the mathematical system known as PA (Peano Arithmetic, which was introduced to formalize key properties of natural numbers) are recursive, i.e. realizable by a computer program, notations satisfying PA ("intended" means that mathematicians think of a fixed model when working with a given theory, even though this theory has other models which differ from the intended one). To bring computational structuralism a little closer to practice (in particular, to situations where computational decisions cannot be postponed indefinitely), we argue that functions which are *primitive recursive* in an intended notation must be the same as standard primitive recursive functions (a primitive recursive function can be thought of as a function that is realizable by computer program which cannot use unbounded loops). Computational structuralism disregards this requirement. To develop our position, we employ punctual structure theory (a subbranch of computable structure theory).

Structuralists distinguish two criteria of structurality: the invariance account (roughly, a property is structural if whenever it holds of something, it holds of anything isomorphic to it) and the definability account (a property is structural if it can be defined in the language of the structure). These accounts face a few challenges in explaining the concept of structurality, particularly of computational properties. To overcome this, our project proposes a hybrid account that allows for relations of structural similarity other than isomorphism. Additionally, the project investigates how computable structure theory can help finding structural characterizations of important computational properties and discusses whether they can be reconcilled with the definability account.

The third area focuses on the question of how to determine the intended model of a theory. We put forward a natural hypothesis: if a theory has an intended model, then it's the simplest one, where the concept of simplicity is clarified using natural complexity hierarchies applicable to structures. Recent results confirm this hypothesis for PA, but we will analyse it further to assess its broader utility.

The fourth area deals with a challenge raised by a famous philosopher Benacerraf: how we acquire knowledge about abstract mathematical structures? The project aims to consider a restricted version of this challenge—the distinguishability problem—which can be formulated as a game between a student and a teacher. The student has access to a model and the teacher, wanting to know whether student's model is correct, asks the student questions about relations that hold in the model he has in mind. A suitable guessing strategy can lead the teacher closer to the truth. The project examines this problem within the context of the learning theory of algebraic structures.

By investigating these four areas, the project aims to contribute to the understanding and development of structuralism in mathematics, providing insights and solutions to the challenges and questions raised within the field.