ON ZARISKI PAIRS OF SURFACE SINGULARITIES

The project deals with the topology of complex surface singularities.

Let g_0 and g_1 be two polynomials in three complex variables z_1, z_2, z_3. We assume that g_0 and g_1 vanish at the origin $\mathbf{0} \in \mathbb{C}^3$ and that the corresponding germs of surfaces, $V(g_0)$ and $V(g_1)$, have an isolated singularity at $\mathbf{0}$. It is well known that if $V(g_0)$ and $V(g_1)$ have the same embedded topology, then they have the same Milnor number, μ. On the other hand, it is quite possible for two isolated surface singularities to have the same Milnor number (or even the same Teissier μ^*-invariant) and distinct embedded topologies. Disproving a conjecture of Yau, Artal Bartolo further showed that the embedded topology of an isolated surface singularity is not determined by the piece of data consisting of the abstract topology of its link and the characteristic polynomial of its monodromy. However, in practice, given g_0 and g_1 with the same characteristic polynomial (or the same monodromy zeta-function), the same Teissier μ^*-invariant and with homeomorphic abstract links, it is extremely difficult to determine whether $V(g_0)$ and $V(g_1)$ have the same embedded topology or not. In this project, we are interested in certain types of surface singularities which are "likely to systematically produce" pairs of germs $V(g_0)$ and $V(g_1)$ sharing all these invariants but having different embedded topologies.

More precisely, let us now assume that for $j \in \{0, 1\}$, the polynomial g_j is of the form $g_j = f_j + z_i^{d+m}$ ($i \in \{1, 2, 3\}$), where m is an integer ≥ 1 and f_j is a non-constant, reduced, homogeneous polynomial of degree d satisfying the following (non-restrictive) conditions:

(i) the singular locus of the projective curve $C_j \subseteq \mathbb{P}^2$ defined by f_j does not meet the coordinate lines $z_i = 0$ ($1 \leq i \leq 3$);

(ii) f_j is convenient (i.e., the Newton diagram $\Gamma(f_j)$ of f_j meets each coordinate axis) and Newton non-degenerate on any face Δ of $\Gamma(f_j)$ if Δ is not top-dimensional.

Then g_j defines an isolated surface singularity of $\mathbb{C}^3$ called a *Lê–Yomdin singularity*. Let us also assume that the projective tangent cones of $V(g_0)$ and $V(g_1)$ (namely, the projective curves C_0 and C_1) make a *Zariski pair of projective curves* in $\mathbb{P}^2$ (i.e., C_0 and C_1 have the same combinatorics and the pairs $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are not homeomorphic). Then we say that $(V(g_0), V(g_1))$ is a *ζ-Zariski pair of surface singularities* if the germs of g_0 and g_1 at $\mathbf{0}$ have the same monodromy zeta-function and if the corresponding links, K_{g_0} and K_{g_1}, have the same abstract topology. Recently, Oka and the author showed that if the singularities of the curves C_0 and C_1 are Newton non-degenerate in appropriate local coordinates, then $(V(g_0), V(g_1))$ is a ζ-Zariski pair of surface singularities such that g_0 and g_1 have the same Teissier μ^*-invariant and belong to different path-connected components of the μ^*-constant stratum. We call such a special ζ-Zariski pair a *μ^*-Zariski pair of surface singularities.* Of course, being a μ^*-Zariski pair of surface singularities does not imply for $V(g_0)$ and $V(g_1)$ to have distinct embedded topologies. However, it is a *necessary* condition for that and an important step in the understanding of the problem.

The first goal of this project is to investigate an analogous question for the μ-constant stratum. More precisely, we are asking whether it is true that the Newton non-degeneracy of the singularities of the curves C_0 and C_1 also implies that $(V(g_0), V(g_1))$ is a *μ-Zariski pair of surface singularities*, that is, a μ^*-Zariski pair for which not only g_0 and g_1 lie in different path-connected components of the μ^*-constant stratum but also belong to distinct path-connected components of the μ-constant stratum — a much stronger statement.

The second objective of this project is to investigate problems similar to those mentioned above (for both μ^* and μ) in the more general situation where (C_0, C_1) is a Zariski pair of *weighted* projective curves in a weighted projective plane.

Finally, our third and last goal is to study analogous problems for some other classes of surface singularities (i.e., singularities which are *not* of (weighted) Lê–Yomdin type).