CONTINUOUS AND DISCRETE OPERATORS IN SPECTRAL THEORY AND HARMONIC ANALYSIS

This research project explores the relationship between discrete and continuous concepts in the fields of spectral theory and harmonic analysis. These two areas of mathematics correspond to two major parts of the project. While being relatively independent from each other, they involve closely related methods and they share motivations from probability theory.

In the spectral-theoretic part of the project, we aim to investigate the properties of certain linear operators that arise in the study of random walks and related continuous-time processes, called *Lévy processes*. These operators, known as *Toeplitz operators* and *Lévy operators*, have an inconvenient feature: they are not *normal operators*. This makes them challenging to analyse using standard methods. Our goal is to understand their structure by developing new mathematical techniques and investigating their spectrum. One of the specific questions that we will try to address is inspired by Widom's conjecture, which provides hypothetical conditions under which the eigenvalues of *Toeplitz matrices* converge to the spectrum of the limiting Toeplitz operator (Figure 1).

The harmonic analysis part of the project focuses on discrete analogues of classical operators in this field, known as *singular integrals* and *maximal operators*. Our programme originates in a century-old open problem, which asks whether the discrete counterparts of the *Hilbert transform* H, the prototypical operator in this area, have operator norms equal to the norm of H. A probabilistic method was invented recently for one such discrete analogue of H. We aim to generalise this technique to other important operators. We will also explore applications of these operators in the theory of stochastic processes. Our study will involve diffusions in a half-plane or a half-space, which are allowed to leave this region only through lattice points (Figure 2).

In addition to these main parts, we have three additional goals. One of them involves studying the geometry of eigenvectors of Toeplitz matrices. Another goal is to explore the concept of *bell-shaped functions* and its applications in probability and spectral theory. Lastly, we aim to construct spaces of functions with specific smoothness properties, motivated by recent advancements in the theory of Lévy operators.

The questions addressed in this project belong to very active areas of theoretical mathematics. We expect that the outcome of the project will significantly broaden our knowledge of spectral theory of non-normal operators, improve our understanding of mutual relations between discrete and continuous harmonic analysis, and possibly also open up new research directions.

Throughout the project, we will investigate how discrete and continuous notions interact, and we hope to provide additional insights into a few long-standing open questions. Although solving these problems is not on the list of objectives of this project, our work aims to contribute to a deeper understanding of them.

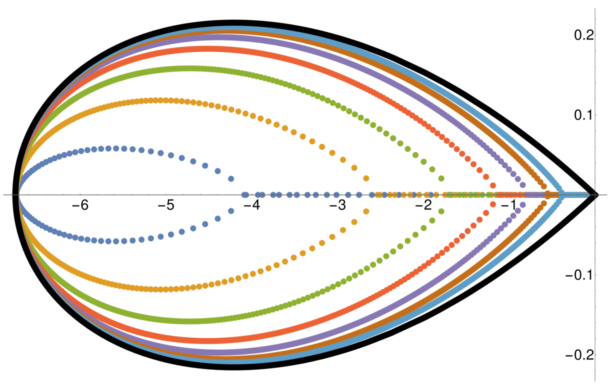

Figure 1. Eigenvalues of Toeplitz matrices with entries

$$a_{i,j} = \begin{cases} 1.1|i-j|^{-1.5} & \text{when } i > j, \\ 0.9|i-j|^{-1.5} & \text{when } i < j, \\ -2\zeta(1.5) & \text{when } i = j \end{cases}$$

of size $100, 200, 400, 800, 1600, 3200, 6400$ (colour points) compared with the expected limit (black line).

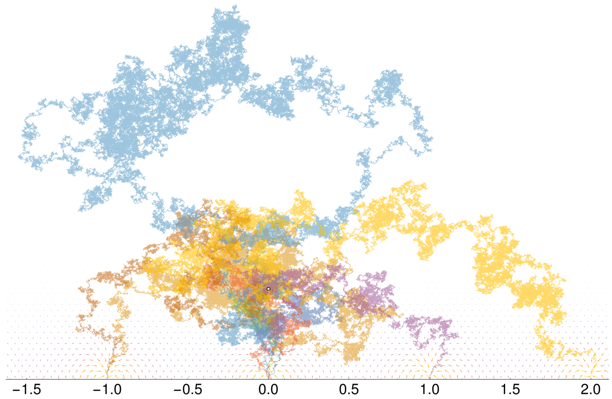

Figure 2. Nine sample paths of the Brownian motion in a half-plane confined to hit the boundary at a point with integer coordinate.