Applications of multiorders and the f-bar pseudometric to actions of countable amenable groups

Abstract for the general public

Ergodic theory and topological dynamics were initially developed for actions of the group of integers $\mathbb{Z}$. Therefore many classical results and proofs use extensively the natural order $<$ on that group; i.e. the "direction of time". Notice that the natural order on $\mathbb{Z}$ is *invariant*, that is, if $n < m$ then for any integer k we have $k + n < k + m$. This invariance allows one to introduce intuitively clear notions of the "future" and the "past', which play a fundamental role in many considerations concerning the dynamics.

One of the difficulties encountered in extending ergodic theory from $\mathbb{Z}$ actions to multiparameter actions of other countable groups, most prominently, countable amenable group actions, is that the sense of "direction of time"–and the resulting mathematics–are lost: In general, there is no invariant order, hence neither the "past", nor the "future" have a definite meaning. Nevertheless, it is still possible to develop a good deal of ergodic theory and topological dynamics for amenable (or even sofic) group actions, as long as a directed time is not used.

Multiorder is a new tool developed by Downarowicz (the PI of the present propsal), Oprocha, Więcek, and Zhang. It is a version of an *invariant random order* (IRO) introduced years back by John Kieffer. Roughly speaking, an IRO is a family of orders $\{<_\alpha\}$ of the group G that is "jointly" invariant, that is, whenever $g <_\alpha h$ $(g, h \in G)$ then for every $k \in G$ we have $kg <_\beta kh$, where α and β are appropriately related (we skip the details of this relation). Multiorder is an IRO satisfying the additional requirement that all involved orders are of type $\mathbb{Z}$, that is they order the group identically as the group of integers $\mathbb{Z}$ is ordered by the usual inequality relation $<$. Thus, multiorder allows to determine the "past" and the "future" of every orbit in exactly the same fashion as it is done for the classical actions of the group $\mathbb{Z}$. In fact, multiorder induces an orbit equivalence to a $\mathbb{Z}$-action with very good properties such as preservation of the past and future, preservation of conditional entropy and many others. In particular, multiorder allows to effectively identify the Pinsker factor (i.e., the maximal factor of zero entropy) in any measure-theoretic action of a countable amenable group.

The f-bar pseudometric is a tool in symbolic dynamics of $\mathbb{Z}$-actions that has proved to be useful, among other things, to characterize topological models of zero entropy loosely Bernoulli systems (i.e., Kakutani equivalent to Kronecker systems). This is a rare situation, when topological models of a certain measure-theoretic class can be identified via purely topological methods.

Our main desire is to merge these two techniques and adapt the f-bar pseudometric (which, by definition, refers to orbits that are ordered along $\mathbb{Z}$) to actions of countable amenable groups. The passage should be possible via ordering the orbits according to the multiorder, or, perhaps, by passing to the associated orbit-equivalent $\mathbb{Z}$-action. Once such an adaptation is successful, we intend to focus on several open problems, starting from very general questions concerning multiorders, then passing to more specific tasks dealing with general applications of multioders in G-actions, finally, we ask questions associated with the applications of the (hopefully adapted) f-bar pseudometric in the context of loosely Bernoulli G-actions.