Mathematics stands on our natural perception of logical reasoning. Nothing makes mathematics precise but its foundational aspects. Modern mathematics implements the theory of sets and first-order logic to assemble a rigid framework for further development and applications. Alas, a fundamental theorem due to K. Gödel- *Gödel's Incompleteness Theorem*- states that no strong enough formal system can provide a ground for answering all possible well-defined problems unless it is inconsistent, i.e., we can draw a contradiction from it. Applying it to mathematics, we must believe that our formalised mathematics is either inconsistent or too weak to answer all the well-defined questions. As we acknowledge its consistency, the latter possibility solely shines in the sky of mathematics.

There have been natural and fascinating problems to which mathematicians have looked to answer. A legendary example of such questions, among many others, is known as *Cantor's Continuum Hypothesis*. G. Cantor discovered that there is always an infinity greater than any given one. In particular, he showed that the size of the real numbers $\mathbb{R}$ is larger than the size of the natural numbers $\mathbb{N}$. In his hypothesis, he conjectured that every infinite subset of the real numbers $\mathbb{R}$ is in a one-to-one correspondence with either the set of natural numbers (the first infinity) or with $\mathbb{R}$ itself. In other words, it states that the size of the real numbers is the second infinity! The problem turned out to be independent of mathematics, i.e., it is neither provable nor disprovable. Nevertheless, attempts to prove or disprove the Continuum Hypothesis initiated several new topics in mathematics and is still one of the strong drives in modern set theory. To show independence, we should provide two arguments; one against the existence of a counter-example and one against that of a proof.

The widely accepted formalism of mathematics, ZFC[1], provides a giant universe of sets to contextualise mathematics. Now, what is this universe? And how can we understand that? What is the role of Cantor's Hypothesis there? It turns out that such a universe is much more complex than other typical structures in mathematics. Denoting this universe by V, Gödel demonstrated that, by shrinking V in a definable fashion, one can obtain an inner universe, denoted L, so that the Continuum Hypothesis holds true in this sub-universe. Therefore, he showed that no one can provide a counter-example to the Continuum Hypothesis while working in ZFC. But it took three decades for mathematicians to demonstrate that no one can also provide proof for the Continuum Hypothesis, again upon ZFC. It was P. Cohen who invented the method of forcing to show that one can expand V to an extended universe V^* so that the hypothesis is false there. Though these two approaches seem opposite, they constitute a heavy and fascinating overlap, which is less understood thus far!

A filter $\mathcal{F}$ on a set X is a means to measure the grossness of a subset of X. So a member of the filter is considered gross, saying it is of measure 1, and a set that lies into the complement of a full-measure set is of measure 0. However, there may be sets of neither measure 0 nor measure 1. We would have a fair understanding of the concept of a measure in mathematical probability. The *Club filter* defined on an uncountable infinity (cardinal) κ is crucial in the Theory of Sets. A fundamental question- relevant to both the internal and external approaches of Gödel and Cohen, respectively- asks how many sets with intermediate values (according to the Club filter) one can find so that the intersection of each two of them is of measure 0, i.e., they are almost disjoint. Generally speaking, the question asks about the "width" of the structure of subsets of κ as measured by the Club filter. The objectives of our project mainly concern the Club filter on the first and second uncountable infinity.

The above-mentioned width has a specific name, the saturation of the filter. I.e., the least infinite number λ, for which there is no λ-sized collection of pair-wise almost disjoint sets of intermediate values. A celebrated theorem states that the statement of the Club filter on the second infinite being $\aleph_2$-saturated to ZFC bears the same strength as adding a *Woodin cardinal* to ZFC. The existence of a Woodin cardinal is a strong form of the *Axiom of Infinity* that impressively impacts our universe of sets. However, the Club filter on the third infinity $\aleph_2$ can never be $\aleph_3$-saturated, but it is widely unknown if a particular restriction of the Club filter on $\aleph_2$ can be $\aleph_3$-saturated.[2]

We shall focus on the latter question and some other relevant technical questions about the saturation problem of the Club filter on the second infinity. An in-depth study of these questions would lead us to discover more fundamental properties of the universe of sets V. We shall use complex methods and techniques in the theory of forcings to tackle our objectives, where we will possibly need to gain a deeper understanding of Strong Axioms of Infinity, e.g. Woodin cardinals, in the theory of inner models and the determinacy of infinite games.

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[1] Zermelo–Fraenkel Set Theory with Axiom of Choice.

[2] The restriction is called the Club filter on $\aleph_2$ modulo a particular set S^1_2.