ERGODIC THEORY MEETS COMBINATORIAL NUMBER THEORY

ABSTRACT FOR THE GENERAL PUBLIC

If we have a sequence starting with $4, 7, 10, 13$, what will be its next term? Questions of this sort are meant to check pattern detection skills; for that reason, they are often used to test mathematical abilities. In the example above, each subsequent term is greater by 3 than the previous one; hence the next term will be 16. Sequences of this form, in which the difference between consecutive terms is constant, are called *arithmetic progressions*.

The search for patterns like arithmetic progressions in subsets of natural numbers is a central task of the subfield of pure mathematics called *combinatorial number theory*. It is known that large subsets of natural numbers contain plenty of "reasonable" patterns, which includes arithmetic progressions. One of the main objectives of the area is to quantify these statements, either by showing that all sets of a certain size contain a pattern or by estimating the number of patterns in sets of a given size.

By contrast, *ergodic theory* examines the statistical behaviour of processes over long time, and its origins lie in the study of large systems of particles in statistical mechanics. The systems to study range from relatively simple ones, such as rotations on a circle, to highly complicated flows on surfaces and their higher-dimensional generalisations. If possible, one wants to obtain a definitive form (e.g. *space average*) for the long-term behaviour of the system (i.e. its *time average*).

On the surface, there seems to be little connection between these two apparently distant areas, both in terms of the scope of research and available techniques: combinatorial number theory tends to utilise quantitative, finitary methods while ergodic theory has a preference for qualitative, infinitary techniques. Yet there is a strong link between these two branches of pure mathematics, forged by Furstenberg's ergodic-theoretic proof of the Szemerédi theorem from combinational number theory. Szemerédi proved that each sufficiently large subset of natural numbers contains arithmetic progressions of arbitrary length; Furstenberg rederived this result by relating it to the problems of multiple recurrence in ergodic theory. The interaction between combinatorial number theory and ergodic theory initiated by Furstenberg has proved enormously fruitful, creating a virtuous cycle that lead to countless new developments in both branches of mathematics.

The goal of this project is to tackle a number of open problems at the interface between ergodic theory and combinatorial number theory. All of them concern the study of long-term qualitative behaviour of *multiple ergodic averages*, which are primary analytic objects appearing in ergodic theoretic problems, and the quantitative behaviour of their finitary analogues showing up in combinatorial number theory. The point is to obtain sufficient knowledge about their behaviours to apply it to specific unresolved questions in both aforementioned areas. Some of the questions that I aim at have ergodic flavour: I want to show that if we take a multiple ergodic average along linearly independent sequences over a "reasonable" system, then in the long term the linear independence of the sequences will induce probabilistic independence of the average, in the sense that the average will converge to the product of integrals of individual functions. I will work on this problem in two settings: for *Hardy field sequences* over systems of *commuting* transformations, and for polynomials over systems induced by *nilpotent* actions. Other questions are combinatorial in nature: I plan to prove new bounds for the size of the largest subset of integers avoiding certain polynomial patterns, addressing a problem posed two decades ago by Gowers. A successful resolution of aforementioned problems, whether complete or partial, will open up new research avenues, answer longstanding problems and fill in gaps in existing knowledge.