Popular description of the project: Mathematics of fluids

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The fluid mechanics from the very beginning has been a source of motivation to develop mathematical theories. An irreplaceable impact one finds in the theory of Partial Differential Equations (PDEs).

Let us start with one of the simplest, in the formulation, equations: the Euler system for incompressible flows

$$v_t + v \cdot \nabla v + \nabla p = 0, \qquad \operatorname{div} v = 0. \tag{1}$$

We look for the velocity and the pressure of the fluid under the incompressibility constraint of the flow. Formally, this system is hyperbolic, but internal (unknown) force being physically the pressure of the fluid is determined by the following constraint

$$-\Delta p = \operatorname{div} \operatorname{div} (v \otimes v). \tag{2}$$

The special structure of the system leads to the global solvability in 2D case for arbitrary large, but smooth, initial configuration. For weak solutions in low regularity the picture is different. Nowadays, the theory developed thanks to the convex integration delivers a large number of non-unique solutions to the Euler Equations. We shall mention that uniqueness is related to the regularity of the solutions. For this reason, currently the mainstream is directed into searching for blow-up phenomena, mostly for compressible version of the systems. This is in particular related to open Millennium Problem concerning regularity of solutions to the Navier-Stokes equations. Although the systems under consideration are well established classical models, the methods are the cutting edge of the current analysis.

Nowadays science finds other origin of systems of type (1), here we have in mind in principle models arising from the collective behavior description based on the Agent-Based approach. In the simplest version they are governed by systems of Ordinary Differential Equations, then by finding their hydrodynamic limits we obtain a large class of fluid-type equations. Such approach is necessary if the number of individuals becomes large. The equations are connected to control of traffic flows of cars on the highways, dynamics of motion of schools of fish, but also opinion in the society. In general in all phenomena where particles/individuals are obeying aggregation-repulsion-alignment interactions. Let us emphasize that hydrodynamical models allow to explain phenomena of aggregation, segmentation or alignment in large groups of individuals.

The project aims at analysis of mathematical problems related to fluid-type systems in the language of PDEs. We put the attention on questions related to the existence of weak solutions, sometimes suitable notion of them, on the other hand we plan to investigate the questions concerning uniqueness and stability of solutions. Finally we want to develop new techniques based on the theory of PDEs on metric graphs in order to create a new approximation of the classical systems. Last, but not least we aim at revision of current techniques based on the Maximal Regularity theory of the parabolic type in terms of the interpolation theory and theory of function spaces of Besov and Lorentz type.