Equivalence in Substructural Logics

Logic is about information processing. Information processing is physical. But classical logic does not care two hoots about physics. Data can be used in any order: if you can get A from B and C, you can get A from C and B. Redundant data can be added: if you can get A from B, then you can get A from B and C. A single datum can be used multiple times: if you can get A from C and you can get B from C, then to get A and B you need only one C, not two copies of it. These features are known as the *structural rules* of *exchange*, *weakening*, and *contraction*, respectively. They are well and good in the abstract, but when one deals with implementations, day-to-day reasoning, or artificial intelligence, they crack.

Sequential access to resources does not satisfy exchange. Order matters. Putting on one's socks and shoes is not the same as putting on one's shoes and socks.

Probabilistic reasoning does not obey weakening. Extra information matters. If Twitty is a bird, then Twitty probably flies. But if Twitty is a bird and Twitty is a very good diver native to the Antarctic, then Twitty probably does not fly.

Contraction can fail, too. Using resources sometimes means using them up. A €5 coin can buy you a cup of coffee *and* a €5 coin can buy you a sandwich. But a €5 coin cannot buy you a cup of coffee *and* a sandwich.

Substructural logics take the physics of information processing seriously. In them, structural rules do not apply, at least not all, not fully, or not always. Substructural logics have been succesfully used in linguistics (Lambek calculus), in approximate and probabilistic reasoning (many-valued logics), in content-sensitive reasoning (relevant logics, non-Fregean logics), in engineering (fuzzy logics), and in computing (intuitionistic logic, linear logic).

Classical logic still reigns, but she is not the absolute ruler any more. Logics are many and varied. They need to be studied, analysed and classified, so that we could put them in appropriate drawers and keep them there for future use. The essence of a logic is given by the *equivalence connective*, typically written $\leftrightarrow$. For if a logic $\mathbb{L}$ cannot distinguish between A and B, then $A \leftrightarrow B$ is a theorem of $\mathbb{L}$. And conversely, if $A \leftrightarrow B$ is a theorem of $\mathbb{L}$, then we can replace A by B in any context and $\mathbb{L}$ cannot tell.

In this project we study the essence of substructural logics—the equivalence connective in them. We use mathematical tools: universal algebra, category theory, proof theory.