The de Rham cohomology of p-group covers

DESCRIPTION FOR THE GENERAL PUBLIC

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Algebraic topology is a field of mathematics that deals with manifolds, i.e. spaces that are nearly "flat" on small scales. For example, the Earth looks flat from our perspective, but in fact it is round. *Algebraic geometry* studies *algebraic varieties*, i.e. sets defined by polynomial equations. The coefficients of these equations might be e.g. real numbers – in this case we obtain a topological manifold. However, those coefficients might also belong e.g. to $\mathbb{F}_p$ (where p is a prime number). $\mathbb{F}_p$ is the set of remainders for division by p, with the operations of addition modulo p and multiplication modulo p. An algebraic variety over $\mathbb{F}_p$ is not a manifold, but various topological techniques still can be applied. *Cohomology* is a classic invariant of manifolds in topology and of algebraic varieties in algebraic geometry. In topology this concept allows to count the "numbers of holes" of a given manifold. There are several different incarnations of this idea. The *de Rham cohomology* measures the extent to which the fundamental theorem of calculus fails on general manifolds. Another cohomology theory is the *Hodge cohomology*. The de Rham cohomology and Hodge cohomology coincide for topological manifolds. However, for algebraic varieties over $\mathbb{F}_p$, those cohomologies can behave in a different way.

In our research we investigate curves (varieties of dimension one) with a fixed set of symmetries. Specifying the set of symmetries of a curve X is equivalent to giving a "nice" map (called *cover* of curves) from X to another curve Y. In a cover there is the same number of points of X over almost every point of Y. The remaining points are called *branch points* (see Figure 1 for an example). In this context we try to describe the structure of the Hodge and the de Rham cohomology of a curve over $\mathbb{F}_p$, accounting for the action of symmetries. In the topological context it is a classical and well-researched topic. However, over $\mathbb{F}_p$ the behaviour of the Hodge and the de Rham cohomology remains a mystery. Our previous result shows that if a variety has at least one point bad enough, then the Hodge and de Rham cohomologies differ from each other. It is somehow a surprising result, since it relates global objects (cohomology theories) to the local behaviour of the variety. The goal of this project is to explain this phenomenon.

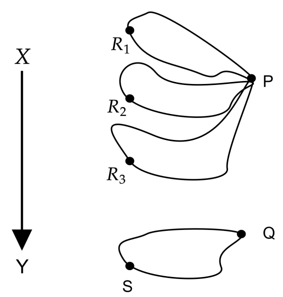

Figure 1: A cover of curves. Q is the only branch point, since only P lies above Q. Over any other point of Y there are three points of X – for example over S there are points R_1, R_2, R_3.

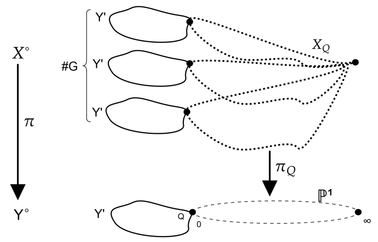

Figure 2: Degenerate cover approximating the cover from Figure 1.

Mentioned results suggest that the de Rham cohomology decomposes as a sum of certain global and local parts. The global part should depend only on the "topology" of the cover (e.g. the number of branch points), while the local parts should depend only on a small neighbourhood of the fixed points of the symmetries. Moreover, the global part of the Hodge cohomology and the de Rham cohomology should be the same. We expect also that the local parts can be described in terms of *Harbater–Katz–Gabber covers*, i.e. covers of the line that are branched only over one point. This follows from the fact that we expect that every cover can be continuously deformed to a trivial cover glued with a bunch of Harbater–Katz–Gabber covers, corresponding to the branch points (see Figure 2 for an example).

In the second part of the project we would like to study various generalizations of this conjecture. In particular, we would like to investigate whether an analogous decomposition of cohomology holds for abelian varieties. *Abelian varieties* are algebraic varieties, whose points can be added. Abelian varieties over $\mathbb{F}_p$ are notably applied in cryptography and for the factorization of large integers.