

SINGULARITIES IN CALCULUS OF VARIATIONS AND PDES

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As a vague and general principle, the observable states of systems we consider in nature are the ones that minimize some energy. The object of calculus of variations is to prove existence of minimizers of such energies and to study the properties of these minimizers trying to describe them as precisely as possible. That is why numerous phenomena in physics, biology, economy and other fields can be described with the help of calculus of variations and partial differential equations (PDEs).

The goal of my research projects is to develop new techniques in calculus of variations and PDEs where, in spite of many results already obtained, several challenging questions remain open. My research program can be divided into three directions intimately related to each others and all connected to the study of singularities. These are points where the function describing a system is not continuous or where its variations become very large. In each direction I consider different problems mainly inspired by three physical problems. The first one is the study of nonlinear elasticity, where one wants to describe the state of an elastic material such as a rubber. The second one is the study of superconductivity which is the loss of resistivity in metal and alloys at very low temperature. I am particularly interested in vortices in superconducting samples. These are analogous to the vortices observed in fluids and are spots where the superconductivity is destroyed. The third physical problem is the study of Coulomb gases which are gases of interacting particles.

The first direction of my research project deals with the study of problems with lack of compactness in the calculus of variations. Roughly speaking I study problems where the existence of minimizers for some energy is not clear and does not follow from the direct method of calculus of variations. Actually the existence of minimizer is not always true in this situation and a careful study of the problem has to be undertaken. An example of such problem is the problem of existence of minimizers for the neo-Hookean energy. This energy is widely used by physicists in elasticity but it is still an open problem to justify mathematically the well-posedness of the model in three dimensions. In this project I plan to study the singularities responsible for the lack of compactness of the neo-Hookean energy.

The second main line of my research concerns the study of the so-called inner variations. They are different from the classical outer variations used to obtain the Euler-Lagrange equations. Only few results on the regularity of solutions to such equations are available as opposed to the many results available for equations obtained via outer-variations. It turns out that the study of the regularity of solutions to such equations are related to the description of vortices in superconductivity and of Coulomb gases. In my research project two regularity problems are proposed on these domains. I also consider inner-variations in the context of nonlinear elasticity and this is related to the justification of the equilibrium equations in elasticity which is an open problem.

The third direction of my research is the construction of solutions to PDEs with prescribed singularities described by points where the solution or its variations become huge. These solutions are obtained by a constructive method and give precise estimates on the solutions. We apply this method to produce new solutions for the Gross-Pitaevskii equation describing moving vortices in a superconducting sample.