PARTIAL DIFFERENTIAL EQUATIONS ON SUBANALYTIC DOMAINS AND MANIFOLDS

Most of the mathematical problems that arise from engineering involve finding the minimum of a functional $E(u)$, generally an energy, where u is a function on a domain Ω of $\mathbb{R}^n$. Such problems generally lead to finding the solution of an elliptic partial differential equation such as:

$$\begin{cases} -\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.1}$$

Here, Δ stands for the Laplace operator $\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x)$, $\partial\Omega$ for the boundary of the open subset Ω, and f is some given function. More generally a typical problem emanating from engineering is to investigate the existence and uniqueness of solutions u of the equation

$$Lu = f \text{ on } \Omega, \tag{0.2}$$

satisfying some boundary conditions, if L is an elliptic differential operator and Ω is a bounded open subset of $\mathbb{R}^n$.

During the seven last decades, an overwhelming number of articles were published to show existence and regularity of solutions when L has sufficiently regular coefficients, f is sufficiently integrable, and the boundary of Ω is sufficiently well-behaved. This kind of results makes it possible to find approximations of solutions using a computer.

The aim of the present project is to carry out the theory of elliptic partial differential equations on any subanalytic domain, with possibly non Lipschitz boundary. Let us recall that subanalytic sets are the subsets of $\mathbb{R}^n$ that are locally the projection of a set that is defined by some equalities and inequalities on analytic functions. In particular, the subanalytic category comprises all the semi-algebraic sets, which are the sets defined by finitely many sign conditions on polynomials. Hence, a concrete example of mathematical problem that we wish to address is

$$\begin{cases} -\Delta u(x) = f(x) & \text{if} \quad P_1(x) > 0, \dots, P_k(x) > 0 \\ u(x) = g(x) & \text{if} \quad \exists\, i \le k,\ P_i(x) = 0, \end{cases} \tag{0.3}$$

where f and g are sufficiently integrable functions, and the P_i's are n-variable polynomials. We here give Dirichlet boundary type conditions for the Laplace operator but the ultimate goal is to investigate all the usual boundary conditions on subanalytic domains for any (strongly) elliptic differential operator with sufficiently regular coefficients.

The advantage of working on any domain defined by polynomial or analytic inequalities is conspicuous, as polynomial constraints naturally arise from engineering problems, such as for instance those met in artificial intelligence or financial mathematics. But the aim of the project is to solve the problems like (0.3) as much as to provide all the necessary material, that can be useful for many other purposes, to do it. This involves to carry out a complete theory of Sobolev spaces of subanalytic domains.