Function spaces for local smoothing

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If I show you a picture of a lake on which you can see every single droplet of water, and if I give you the mathematical equations which describe the behavior of waves in the water, can you tell me exactly what the lake looked like one minute after I took the picture?

Mathematics tells us that, in principle, the answer is yes. However, such a calculation would be unfeasible in practice, and it is not possible to take a picture with the required level of detail. Moreover, for most applications it is not necessary to predict exactly where every single droplet of water will be. And, in fact, the movement of water in a lake is just one basic example of an important class of phenomena which also involves sound, radio, electromagnetic, and even gravitational waves.

So, instead of trying to make perfect predictions based on perfect knowledge about one specific phenomenon, mathematicians attempt to make slightly more limited predictions based on slightly less detailed information about a general class of phenomena. The simplest example of this is the principle of conservation of energy from physics: if I tell you how much energy is contained in a certain system of particles which is closed off from the rest of the world, then the total energy in the system will be exactly the same after one minute, even if the particles in the system might have rearranged themselves.

Mathematicians are also interested in other quantitative properties of physical systems. For example, we might be interested in the height of the highest wave in our lake, and such a quantity is typically not preserved. So, instead, mathematicians merely try to estimate such a quantity. It is then important that the water in the lake is not too chaotic, and that there are no waves smashing powerfully into each other. If the water in the picture is nice and smooth in this way, then it turns out to be possible to produce good estimates for a given quantitative property of the water. In fact, in 1991, mathematicians were able to determine exactly how smooth the lake needs to be, to estimate a given quantity after exactly one minute. And they were able to do the same for a much more general class of phenomena which involve waves.

However, having solved that problem, a surprising observation was made in 1993. Suppose that, instead of wanting to estimate a given quantitative property after exactly one minute, you only want to estimate the average of the quantity during that minute. Then the lake doesn't have to be quite as smooth on the picture as when you want to get estimates at a certain fixed time. This principle is called *local smoothing*.

Unfortunately, it has been an open question since 1993 exactly how smooth the water needs to be on the picture if you want to estimate the average of a given quantity. And this open question is linked to a variety of other, seemingly completely unrelated and yet very important, open problems in mathematics.

The project will use a new approach to local smoothing, by keeping track of information in the picture which is typically neglected. In recent work by the principal investigator it was shown that, by keeping track of the directions in which the waves on the picture are moving, one needs less smoothness to make good predictions. The project will develop this approach further, in several ways.

For example, it will be determined to which properties of the water in the lake this new approach can be applied. When does this new approach improve the existing estimates?

On the other hand, suppose we replace the lake itself by something more complicated, such as water flowing through an erratic river in the mountains. Can we still use information about the directions in which waves move to produce good estimates?

The project will also introduce different ways of using information about the directions in which the waves move. Do these form the basis of even more efficient ways to make predictions?

The project is expected to answer these and other questions, and thereby significantly improve our understanding of the principle of local smoothing.