

Turán-type problems in graph theory

Many objects and physical phenomena can be modeled using a graph: a collection of points (called vertices) some of which are connected by an edge. For instance, one can use graphs to model road connections between cities or social media sites like Facebook, where vertices are all users and edges represent connections between friends. On such graphs one can use mathematical methods that allow easier analysis of the network's properties and more effective construction of algorithms. This, in turn, allows obtaining particular wanted information such as the shortest route between cities or the speed at which information is spread among people. The simplicity and generality of graphs, as well as their wide practical applications, force a strong need to develop new techniques of dealing with graph theory questions and led this field to become an important area of research in mathematics and computer science.

One of the most significant results in graph theory is the Turán's theorem from 1941 determining the maximum possible number of edges in a graph on a fixed number of vertices which does not contain a complete graph of a given size. The project deals with various generalizations of this theorem in which one asks not for the number of edges, but for the number of other subgraphs, induced subgraphs or minimum degree (minimal number of incident edges to a vertex), as well as with similar questions stated for oriented graphs (graphs with orientation on each edge) and graphs with colors on edges. Such generalizations are motivated by the recent intensive developments and important open problems in graph theory. The goal of the project is to provide solutions to some of the significant conjectures in the field, investigate the behavior of the considered maximum in various unknown cases, and develop new techniques of dealing with such Turán-type problems.

Among considered generalizations is the maximization of the number of copies of a given graph instead of the number of edges. In this case we do not have global theorems and solving the problem even for small particular graphs is hard and significant. For example, the Erdős conjecture on the maximal number of pentagons in graphs without triangles was solved by the PI after being open for almost 30 years, while the problem on the maximal number of triangles in graphs without two triangles sharing an edge is equivalent to the open problem of obtaining tight bounds for the so called triangle removal lemma, which has important implications outside graph theory. In the project it is planned for instance to prove a general conjecture giving the asymptotic answer in substantial number of cases and to find the maximum for some interesting graph classes like cycles, generalizing the mentioned Erdős conjecture, as well as for directed cycles in the setting of oriented graphs.

The planned methodology involves in particular novel analytic methods based on graph limits and computer assistance. Such methods have been greatly developed in the recent years and have made advances not only in graph theory, but also in statistical physics and data analysis. The PI contributed to this development for instance by settling, together with his collaborators, a general conjecture stated by László Lovász, the recipient of this year's Abel Prize, the most prestigious award in mathematics. Further studies and development of such methods within the project will be of a significant importance and of a great interest.