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Applications of multiorders and tiling systems to studying measure-theoretic and topological actions of amenable groups

Intensive studies of classical dynamical systems (iterates of a single transformation) in the last century bore fruit in the vast amount of theorems concerning recurrence, asymptotic statistical properties of orbits, and chaos. The natural way of extending the theory developed over the last decades is to generalize those theorems to the case with several commuting transformations or even non-commutative groups of transformations. There is a class of groups called amenable groups which are particularly well-suited for such generalizations. Many theorems proven for the classical dynamical systems rely on the property that the group Z of integers is naturally and linearly ordered by the inequality $<$, and this order is preserved by adding any constant, i.e. $n < m$ implies that $n + k < m + k$ for every integer k. Moreover, for any integers $n < m$, the order interval $\{k \in \mathbb{Z} \ : \ n \le k \le m\}$ is finite. In general, amenable groups do not need to admit an invariant order with finite order intervals. In fact, even one of the simplest examples of a commutative group, namely $\mathbb{Z}^2$, does not admit such an order. Henceforth, the results known for classical dynamical systems, cannot be directly transferred to systems with actions of amenable groups, unless they are appropriately adapted and their proofs use new tools specific for more general setup. One of the main goals of the project is to develop such tools. For instance, in classical dynamical systems, there is a notion of a remote past refering to the infinitely distant iterates of the transformation. But how to define such an object for actions of general groups, if it is not even clear how to do it in the fairly elementary case of $\mathbb{Z}^2$? In classical dynamical systems one considers also so-called asymptotic pairs, defined as pairs of points whose orbits approach each other as time advances. Again, how to define an analagon of asymptotic pairs in a generalized dynamical system, where the "time" is a group in which the term advancing has no definite sense? One of the most recent achievements towards overcoming those obstacles is the invention of a multiorder. A multiorder on a countable group is a family of linear orders of the group G, each having finite order intervals, such that although none of these orders is invariant, the whole family is invariant in the following sense: if $\prec$ is an order in this family and $g \in G$, then $\prec'$ defined by $a \prec' b \Leftrightarrow ag \prec bg$ also belongs to this family. Once an order $\prec$ from the multiorder is fixed, the remote past can be defined along $\prec$. Likewise, one can naturally define asymptotic pairs with respect to $\prec$. These (very rough and incomplete) examples give a flavour of how nonstandard tools can be used to generalize some classical notions. We hope that with the help of multiorders and similarly nonconventional concept we will be able to define and study chaos, asymptotic pairs and related notions in generalized dynamical systems (with actions of groups or semigroups of transformations).

One of the key parameters of a dynamical system is its entropy (reflecting the exponential rate of complexity of the collection of orbits of increasing lengths). In the 70s of the 20th century, this notion has been successfully generalized to the case of dynamical systems with actions of commutative groups and soon later to actions of amenable groups. In the classical case, there is a strong connection between positivity of entropy and the existence of chaos and asymptotic pairs in a given system. Analogous results for the actions of amenable groups exist only in partial forms and for a restricted class of groups (those which admit an invariant order). One of possible applications of multiorders is finding a good definition of chaos and proving analogous relations between entropy, chaos and existence of asymptotic pairs in full generality.

The main scientific aim of this project is to expand the research of dynamical systems with the actions of countable amenable groups, using multiorders and tiling systems (a tiling system is another nonconventional tool which has already proved very useful in the analysis of generalized dynamical systems). One of the objectives is to generalize several classical theorems connecting various notions of chaos, entropy and related concepts to systems with actions of countable amenable groups, and possibly obtaining novel results (unknown even for classical dynamical systems) which are still waiting to be discovered.