Generalizations of the graph coloring problem in graphs with forbidden structures

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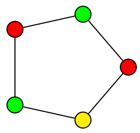
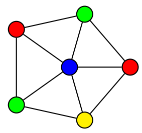
The k-coloring problem is, arguably, one of the best studied and well-known graph problems: we ask whether we can assign k colors to the vertices of a graph in a way that no two adjacent vertices receive the same color. Below there are two examples of graph colorings, using, respectively three (left) and four (right) colors. It is straightforward to check that these graphs cannot be colored using, respectively, two and three colors.



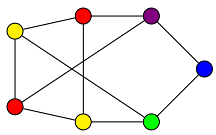


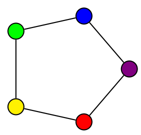
If we have $k \geq 3$ colors, the k-coloring problem is NP-complete. This means, in particular, that we do not know any algorithm which decides whether the graph G on n vertices can be colored using k colors, and does that in time polynomial in n.

However, it may happen that we are interested in solving our problem only for some particular graphs G, with some additional properties (e.g., planar graphs, or such that each vertex has only four neighbors). The intuition suggests that with these additional information we may be able to solve our problem faster. In many cases this happens to be true, and in extreme ones, this may cause our problem to become even trivial. For example, we do not know any algorithm which solves the 4-coloring problem in polynomial time in general graphs G, but if we assume that G is planar, there exists an algorithm working even faster: the one which returns a positive answer for each graph G. Indeed, it follows from the famous Four Color Theorem that each planar graphs can be colored using only four colors.

Therefore, we are interested in theorems which characterize the structure of graphs which belong to some given graph class, since usually they show a way to use our additional knowledge about G to reduce the problem to a simpler one. This, in turn, obviously helps to design effective algorithms. Moreover, these structural theorems usually can be used also to solve more general problems than graph coloring.

In our project we are in fact interested in investigating a generalization of graph coloring, the so-called graph *homomorphisms*. Let G and H be graphs. We say that a function f which maps the vertices of G to the vertices of H is a homomorphism, if from the fact that the vertices u and v are adjacent in G it follows that $f(u)$ and $f(v)$ are adjacent in H. Informally, consider the example below: it shows a homomorphism from G (left) to H (right). If we interpret vertices of H as colors, then each each pair of adjacent vertices in G must be mapped to a pair of adjacent vertices in H.





For a fixed graph H, in the homomorphism problem we ask whether there exists a homomorphism from a given graph G to H. Observe that if H is the complete graph on k vertices, then we obtain a problem which is equivalent to the k-coloring problem. In general, there are no known algorithms to solve these problems significantly faster than by a “brute force” (i.e., by enumerating all possibilities), if G is an arbitrary graph. Hence, we are interested in solving the homomorphism problem in some restricted graph classes — in particular ones that can be defined by forbidding certain structures.