DESCRIPTION FOR THE GENERAL PUBLIC

L-functions are a kind of generating functions associated with some arithmetic, geometric, or algebraic objects. Usually they are defined as Dirichlet series or Euler products which are convergent on a certain half-plane, admit a meromorphic continuation to the whole complex plane and satisfy a Riemann-type functional equation. Historically, they first appeared in classical works by Euler, Dirichlet and Riemann. Probably the best known example of an L-function is the Riemann zeta function. Thanks to seminal works by many famous mathematicians such as R. Dedekind, E. Artin, E. Hecke, and R. Langlands, to mention just a few of them, L-functions became important tools in the study of arithmetic problems, and now occupy a central position in modern number theory.

In 1989 A. Selberg gave an axiomatic definition of a class of functions, nowadays called the *Selberg class* and denoted by $\mathcal{S}$, which conjecturally contains all L-functions of number-theoretical importance. At present, a precise description of elements of $\mathcal{S}$ is known in a very restricted range only. The main conjectures in this direction predict that the degree of every L-function in $\mathcal{S}$ is a non-negative integer (*the Degree Conjecture*), whereas all L-functions of integer degree come from automorphic representations of arithmetic groups (*the General Converse Conjecture*). It is relatively easy to describe the structure of $\mathcal{S}$ for degrees $d < 1$. This follows from some classical results by H.-E. Richert and S. Bochner from the 1950s, and it was done by B. Conrey and A. Ghosh in 1993. Subsequent progress was made by J. Kaczorowski and A. Perelli who confirmed both the Degree Conjecture and the General Converse Conjecture in degrees less than 2. In a recent paper (2020), which is yet unpublished, they managed to describe L-functions of degree 2 and conductor 1 completely, confirming the General Converse Conjecture also in this case.

The principal aim of this research project is to make further progress in the study of $\mathcal{S}$. The basic target would be degree 2 L-functions of conductors greater than 1. A well-known theorem of A. Weil suggests that it is reasonable to study them using twists of L-functions by Dirichlet characters. There are many important problems related to this that need to be clarified. For instance, it is unknown in general how the basic invariants of L-functions from $\mathcal{S}$ behave under such twists. In particular, supposing that a twist F^χ of an L-function $F \in \mathcal{S}$ belongs to $\mathcal{S}$ as well, what is its degree? One can conjecture that the degree does not change under such a twist, but a precise proof of this fact is lacking at the moment. Similar problems can be formulated with respect to other invariants such as the conductor or the root number. Guided by the classical examples of L-functions associated with Hecke and Maass modular forms one can formulate plausible conjectures in this respect. We intend to address these conjectures, as well as some similar related problems, in the current research project.

Parallel lines of research in this project shall focus on the theory of value distribution of L-functions and their applications in quantitative theory of factorizations. Values of the Riemann zeta-function and its generalizations at arguments lying in the critical strip are among the key subjects in analytic number theory. The distribution of values of these functions has been investigated by many mathematicians in the last decades, but still, our current knowledge on this topic is quite limited. Our research in this direction will cover various generalisations of the classical zeta function, where there are many interesting open problems left to solve. The second mentioned line of research shall address factorization properties in commutative monoids. Monoids with the simplest structure, i.e. the free ones, are precisely those with unique factorization. We will study non-uniqueness of factorization in analytic monoids and how they behave in constructions, where new analytic monoids are built from others. We will also attempt to tackle the question of factorization properties of one of the most intriguing monoids: the Selberg class itself.

L-functions are of fundamental importance in number theory and algebraic geometry. There are also other types of zeta functions used in various branches of mathematics (algebra, combinatorics, dynamical systems, etc.). This links the subject of the proposed project to other parts of mathematics and makes research in the subject so attractive.