Combinatorics and structure of Hecke–Kiselman algebras

abstract for the general public

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For any finite set of letters $\{x_1, \dots, x_n\}$ let us consider the set of all words formed from these letters, that is formal strings of the form $x_{i_1} \cdots x_{i_n}$, with the empty word denoted by 1. In this set there exists a binary operation, which to any two words w and v assigns the word wv. We assume that for any word $w1$ and $1w$ is just w. This operation, called concatenation, is associative. The set of words with concatenation forms so called free monoid generated by $x_1, \dots, x_n$.

In the free monoid one can additionally identify certain different words in such a way that under these identifications concatenation remains well defined and associative. That is we impose generators and relations between these generators. In this case we say that the monoid is given by a presentation. For instance, the monoid given by the presentation $\langle x : x^2 = x \rangle$ consists of two elements $1, x$ and multiplication in this monoid is determined by $x1 = 1x = x$ and $xx = x$.

Such constructions of various algebraic structures arise in a very natural way in many areas, including combinatorics, geometry, topology and algebra. One of the examples is the braid group (and braid monoid), which was firstly introduced as an algebraic formalization of braids consisting of a fixed number of strings.

The aim of the project is to study combinatorial aspects of certain class of monoids given by a presentation related to braid relations and investigate the structure of algebras associated to them. We focus on Hecke–Kiselman monoids, introduced as a generalisation of two other families of monoids with similar algebraic properties. This generalisation is given by presentations associated to combinatorial objects called digraphs. A digraph consists of a finite set of vertices, depicted graphically as dots, and a set of edges, depicted as lines or arrows between vertices. Every edge connects two different vertices and can be either unoriented, when it has two symmetric endpoints or oriented, if one vertex is the beginning and second is the end of edge. Hecke–Kiselman monoid associated to such a digraph is given by generators corresponding to vertices, such that $x^2 = x$, and relations between any two generators depending on the edges in the graph. In particular two-element monoid from the example at the beginning is the Hecke–Kiselman monoid associated to the graph consisting of exactly one vertex.

This class of monoids is related to the semigroup analogue of famous Coxeter groups that grew out from the study of reflections. Hecke–Kiselman monoids occur also in the representation theory of finite dimensional algebras, that is in the theory where algebraic structures are studied via representing their elements as matrices. Moreover, because of their connections to braid monoids, some Hecke–Kiselman monoids can be represented diagrammatically.

Various properties of Hecke–Kiselman monoids in terms of the shape of digraphs will be investigated during the project. The first part of the project focuses on combinatorial aspects of monoids associated to graphs with oriented edges. We plan to calculate Gelfand–Kirillov dimension of Hecke–Kiselman algebras. This notion is one of standard invariants used in the study of finitely generated algebras. Roughly speaking, for monoid algebras associated to an infinite monoid, it measures the complexity of the monoid. Secondly we will investigate so called Green's relations for Hecke–Kiselman monoids.

Moreover, classical finiteness conditions of arbitrary Hecke–Kiselman algebras will be studied. We will analyse the ascending chain condition on ideals, which has deep consequences for the structure and representations of the algebra. If it is satisfied, algebra is called Noetherian. We plan to characterize Noetherian Hecke–Kiselman algebras, already described for oriented digraphs, in the more general case. For a monoid defined by a presentation, it is in general a challenging problem to decide whether it consists of finitely many elements. In the case of Hecke–Kiselman monoids only partial results are known. Characterization of finite monoids in more general cases is also planned.