Around the Karlsson-Nussbaum conjecture

Description for the general public

Nonexpansive mappings i.e 1-Lipschitz, similarly like isometries and contractions, form one of the most basic class of nonlinear mappings. Interesting and at the same time very intriguing issue is considering dynamics of such mappings relative to different metric spaces.

For example well known Banach fixed-point theorem describes the behavior of orbits of contractions in complete metric spaces. In other words, if f is a contraction, then there is exactly one fixed point of f such that for all x the sequence of iterates $x, f(x), f(f(x)), f(f(f(x))), \ldots$ converges to this fixed point. What can be said about the behavior of nonexpansive mappings? Nonexpansive mappings are borderline case in the theory of contractions when the Lipschitz constant converges to 1. Moreover their dynamics are much more complicated. When we are dealing with the nonexpansive mapping, additional assumptions are necessary to guarantee the existence of a fixed point. Moreover, even when there is such point of f, the sequence of iterates f^n generally does not converge to the fixed point. Confirmation of the complexity of dynamics of nonexpansive mappings is the fact that researches of the asymptotic behavior of these mappings are one of the most frequently conducted in nonlinear analysis. Previous research has been conducted mostly in Banach spaces, in CAT(0) spaces or in hyperbolic metric spaces.

One of the most important theorem which relates to dynamics of nonexpansive mappings is the Wolff-Denjoy theorem. In classical version it states that if $\Delta \subset \mathbb{C}$ is the unit disc on the complex plane and the holomorphic mapping $f : \Delta \to \Delta$ is fixed-point free, then there is a point on the unit circle ξ such that for all x the sequence of iterates $f^n(x)$ converges to ξ. The above theorem is the subject of interest of many researchers and over the years it was generalised in different directions among others on strictly convex sets in $\mathbb{R}^n$ relative to the Hilbert metric or on strictly convex sets in $\mathbb{C}^n$ relative to the Kobayashi (pseudo)metric.

Interesting class of metric spaces, in which every two points can be connected by a curve with a length equal to a distance between these points, are geodesic metric spaces. Due to these kind of metric spaces it is possible to consider the problem of dynamics of iterates of nonexpanssive mappings in more unified way.

Going a step further in considerations of the dynamics of nonexpansive mappings we talk about the Karlsson-Nussbaum conjecture, which is the main point of this project and at the same time the main only partially resolved problem in this field of research. The Karlsson-Nussbaum conjecture states that that in finite dimensional Hilbert's metric spaces there exist a convex subset of a boundary of convex set such that accumulation points of the orbits of nonexpansive mappings without fixed point lie in this boundary set. Substantiation of the above conjecture and formulation of its counterpart would lead to significant progress in research of nonlinear analysis.