Arithmetic and dynamical properties of sequences given by substitutions

Lambert proved in 1761 that the number π is irrational, and so its digits in the standard decimal expansion do not repeat periodically. Does every digit occur infinitely many times? Everyone believes that it does, simply because there should be nothing special about the decimal expansion of π, no patterns in its expansion were ever observed, and we expect that it's 'random'. Of course, it is not strictly speaking random, since the digits are determined without any random process, but possibly 'quasi-random', meaning that in many respects they behave like a random sequence. For example, we expect that 2 is followed by 1 about as often as it is followed by 3, and that somewhere in the expansion there are arbitrarily long blocks of consecutive occurrences of the digit 7. However, such statements are usually extremely hard to prove, and in the case of π it is not even known whether every digit occurs in its expansion infinitely many times; the same is the case for $\sqrt{2}$ or any other irrational algebraic number. Similarly, it is generally believed that numbers or sequences which are simple in one base, say, powers of 10, should have complicated (or quasi-random) expansions in another (multiplicatively independent) base. However, such a statement needs to be taken with a grain of salt; after all, a power of 10 is even, so the final digit in its binary expansion is zero; similarly, if a real number has periodic decimal expansion, then it is rational, and so its expansion in any base will also be (ultimately) periodic. In other contexts, there will be similar 'obvious' relations; the expectation of a quasi-random behaviour means that there should be no 'hidden' non-obvious relations.

Fortunately, not all problems concerning mutual relations between expansions in two bases are hopeless, and many important results have been obtained. The celebrated Cobham's theorem is concerned with automatic sequences, whose terms are produced by a finite procedure (a finite automaton) from digits of natural numbers in a given base. Cobham's theorem says that any sequence that can be produced in this way using two (multiplicatively independent) bases (for example the decimal and binary base) is necessarily very simple (ultimately periodic).

The aim of this project is to further develop this study for particular classes of sequences. The project lies at the intersection of dynamical systems, combinatorics, and number theory. Our approach to the problem will be founded on the extensive and systematic use of tools from the theory of dynamical systems. We will be interested in various classes of sequences generalising automatic sequences, whose definition is relative to a given base (although the notion of base needs to be interpreted quite broadly here). Of particular importance will be substitutive sequences (for which the notion of base still makes sense, but is no longer an integer). We divide our research into the following tasks:

A. Studying the problem of how to recognise that a substitutive sequence is automatic (proposed recently by Allouche, Dekking and Queffélec).
B. Generalising Cobham's result for automatic sequences to the setting of finite-rank S-adic dynamical systems (with a particular emphasis on constant length S-adic dynamics).
C. Studying Bell's conjecture concerning sequences with finite k-dimension.
D. Generalising Cobham's theorem for k-regular sequences to the wider class of generalised k-regular sequences.

We hope that our research will discover new relations between various objects defined in purely arithmetic and dynamical contexts. In particular, we hope to emphasise the links between S-adic dynamics and extensions of the notions of automaticity and regularity.