Arithmetic of Differential Equations

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A theme that unites various directions of this project is the notion of *periods*, which loosely speaking are mathematical constants which are described in terms of geometry. For example, the number $\pi = 3.1415...$ is a period, since it describes the relationship between the area of a circle and its radius. In contrast, the number $e = 2.71828...$, often defined in terms of properties of the exponential function, is likely not a period, though proving this is beyond reach of current methods of pure mathematics.

More precisely, periods are volumes of regions in Euclidean space described by polynomial inequalities with rational coefficients. More exotic numbers which are periods are the *integral zeta values*, defined as an infinite sum

$$\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k} + \cdots \quad \text{for } k = 2, 3, 4, \ldots$$

though this fact is not at all clear from this description. Euler proved that the values $\zeta(2k)$ are rational multiples of π^{2k}, but very little is known about zeta values at odd integers. In 1978 Roger Apéry showed that $\zeta(3)$ is not a rational number, and still we don't even know whether $\zeta(5)$ is irrational. (Conjecturally, there are no algebraic relations among the odd zeta values and π.) Analogous result for the p-adic value $\zeta_p(3)$ is known only for primes $p = 2$ and 3, which shows a need in developing p-adic techniques in the theory of periods.

Period functions are families of periods that depend on one or several parameters. They satisfy linear differential equations called *geometric* or *of Picard–Fuchs type*, which arise from a construction in algebraic geometry called the Gauss–Manin connection. For example, the elliptic integral

$$\phi(t) = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-t)}}$$

is a period function related to the Legendre family of elliptic curves $y^2 = x(x-1)(x-t)$.

One of the main goals of this project is to study a class of periods (Frobenius constants) arising as matrix entries of the monodromy representations attached to geometric differential equations. We are working on a theory which gives an almost 'canonical' way to write Frobenius constants as integrals. More precisely, they are periods of a particular kind: iterated integrals. This notion has a distinguished role in representation theory of the topological fundamental group of the complex plane with several points removed.

We also study p-adic properties of period functions and, more generally, develop new p-adic methods in the study of periods. Geometric differential equations are known to have a strong property, called the *Frobenius structure*, which often gives rise to interesting p-adic constants. Recent numerical experiments suggest certain uniform behaviour of these constants when p is varying.

This project brings several novel ideas into the study of periods. By developing them, we expect to push further this field of research. In this proposal we tie together classical subjects, motivated by arithmetic geometry and algebraic topology, with some curious arithmetic examples arising recently in the literature. We thus expect that our work will have broad applications in number theory and combinatorics.