Extension property and function theory methods

(Description for the general public)

A celebrated theorem of H. Cartan asserts that if Ω is a pseudoconvex domain in $\mathbb{C}^d$ and $\mathcal{V}$ is a holomorphic subvariety of Ω, then every holomorphic function on $\mathcal{V}$ extends to a holomorphic function on Ω. It is not true, however, that every bounded holomorphic function on $\mathcal{V}$ necessarily extends to a bounded holomorphic function on Ω. It is even rarer for every bounded holomorphic function to extend to a bounded holomorphic function of the same norm, and when this does occur, there is a special relationship between $\mathcal{V}$ and Ω. The problem of describing this relationship was formulated over 60 years ago by Walter Rudin and has two natural motivations. One is provided by spectral sets (e.g. von Neumann sets for d-dtuples of commuting operators in Hilbert spaces), the second by Neveanlinna-Pick interpolation problems. Tautologically there is some holomorphic subvariety $\mathcal{V}$ on which all minimal norm solutions coincide, but sometimes one can actually say something descriptive about $\mathcal{V}$. If $\mathcal{V}$ had the extension property, one could split the analysis into two pieces: finding the unique solution on $\mathcal{V}$, and then studying how it extends to Ω.

There is one easy way to have a norm-preserving extension. We say $\mathcal{V}$ is an analytic *retract* of Ω if there is a holomorphic map $r : \Omega \to \Omega$ such that the range of r is $\mathcal{V}$ and $r|_{\mathcal{V}}$ is the identity. If $\mathcal{V}$ is a retract, then $f \circ r$ will always be a norm-preserving extension of f to Ω. In 2005 Agler and McCarthy published in Annals of Math. a result that if Ω is the bidisk, that is basically the only way that sets can have the extension property. This paper, a breakthrough in the field, was based on Operator-Theory methods which have no chance to generalize to other domains or at least to higher dimensional polydiscs.

On the other hand, as proved by Agler, Lykova and Young in 2019, there is a domain for which there are other ways of constructing extension property sets. This is the symmetrized bidisk, the set $G := \{(z + w, zw) : z, w \in \mathbb{D}\}$.

In recent years a huge progress on the solution to Rudin's question was made. Nevertheless, the problem still requires further deep studies and provides many open questions and conjectures. This is what inspires and motivates the current project.

Some of the tasks proposed in the project seem to be within a range of a good PhD-student. Others are probably more difficult. However, even partial solutions of them will be of great interest and will surely contribute to the theory.

To solve the problems posted in the project, methods coming from complex analysis and functional analysis will be needed. Some of them are standard, some need to be refined and some, we hope, will be developed during the project. Most of the tasks are of pure-theoretical nature. Nevertheless some problems can be overcome with the help of computer-assisted proofs.