A popular summary of the research project

Logico-philosophical foundations of geometry and topology

Geometric reflection on space has accompanied mankind since ancient times. First, in Mesopotamia and Egypt, it had a pragmatic background. It helped the engineers of that time to erect buildings and to preserve the surface of fields flooded by the waters of the Nile River. As far as we know, there was little interest in the theory itself in separation from the real world.

Later, ancient Greeks elevated geometry to the rank of a subject worthy of research for its own sake. What became important was no longer just what geometry could be used for, but above all what can be said about the nature of and mutual connections between objects such as points, lines or figures. With geometry as pure science, the axiomatic method was born, the essence of which is to prove complex facts based on possibly simplest and certain assumptions called axioms.

The best testimony to the method as it was used by the Greeks can be found in Euclid's *Elements*. Without any major changes—bearing witness to the great perspicacity of the Ancients—Euclid geometry reached the XIXth century AD. Only then, mathematicians such as János Bolyai, Carl Friedrich Gauss and Nikolai Lobachevsky discovered geometric systems so revolutionary that they deserved to be called *non-Euclidean*.

Several decades earlier, in the XVIII century, another great mind, Leonhard Euler, had provided the answer to the question: *can you walk all over the Königsberg by passing through each of its seven bridges over the Pregol River only once?*, and thus had laid the foundation for graph theory and, even more importantly for us, *topology*.

Geometry, the basics of which we learn from the early stages of school education, concerns such properties of space as distance, shape, size or similarity of its pieces. Investigations always concern rigid objects: we can rotate and scale them, but we cannot bend them. It is as if they were made of completely inflexible material. Topology, on the other hand, examines those properties of space and its fragments that involve bending, squeezing and stretching. Colloquially speaking, topology is the theory of rubber objects.

After the discoveries of topology and non-Euclidean geometries, the two fields were no longer treated only as tools to unlock the mysteries of space. They were turned into objects of study of philosophers', logicians' and mathematicians'. Questions were asked about the nature of theories themselves, their logical structure, conceptual base, axioms and philosophical foundations.

Both geometry and topology are based on the notion of *point*, naïvely understood as referring to dots of microscopic size. This crucial notion, seemingly innocent, began to bother philosophically-minded logicians and mathematicians. Some of them, like Bertrand Russell or Alfred N. Whitehead, constructed theories of space deprived of the notion of *point* understood as an elementary ingredient of space. Instead, points were assembled out of its chunks. In this way, the first three decades of the previous century witnessed the beginning of the so-called *point-free* geometry and topology.

Around the same time, an American logician Clarence Irving Lewis began to work on systems of logic based on the notions of *possibility* and *necessity*, the so-called *modal logics*. Treated with suspicion, as not equipped with any convincing interpretation, modal logics found its alley in topology. In the 40s of the previous century, Alfred Tarski and John McKinsey demonstrated that the notions of *possibility* and *necessity* have an elegant meaning in topological spaces. They proved that a certain system of modal logic is—in a precise mathematical meaning—a logical scaffolding for some topological spaces.

This project follows the steps of Russell and Whitehead, as well as those of Tarski and McKinsey. On one side, it aims to find a modal logic that lies at the foundation of one of the most elementary systems of geometry based on the relation of *betweenness* and having just one dimension. On the other side, it investigates properties and consequences of topological theories whose points are not taken for granted but are constructed. Still another side of the project is dedicated to examining how two cognitive abilities of human beings: idealising and abstracting, influence the expressive power of different systems of geometry and topology.