Educational project summary

An amazing property of the laws of nature is the fact that we can describe simple models built on the basis of these laws using analytical formulas. We say that such models are solvable. This is because we have conservation laws among the laws of nature. In the case of models described by ordinary differential equations, the conservation laws are the first integrals. Some of these conservation laws are universal. For example, the law of conservation of energy has such a universal character. However, even in the case of simple models, the universal conservation laws are not sufficient to describe them analytically. For instance, to solve the Kepler problem and the two-body problem, it is not enough to know the laws of conservation of energy, momentum and angular momentum. Additional first integrals for the Kepler problem are defined by the Laplace–Runge–Lenz vector. However, the fact that these integrals exist is not at all obvious.

The research in the project is devoted to the problem of how to find such „non-obvious" first integrals in models of which the evolution can be described by ordinary differential equations. Systems of differential equations can be found in almost every field of science — starting from astronomy, physics, mechanics, chemistry, mathematics financial, and ending with biology or ecology. If the tested model has a sufficient amount of first integrals to describe it in an analytical form, we say that such a model (or the corresponding system of differential equations) is integrable. The first results concerning the existence of first integrals and integrability can be found in Newton's famous Principia. Since then, many great mathematicians and mechanics have focused their attention on finding explicit solutions of differential equations that describe known physical systems.

There is a long history of finding tools to distinguish integrable systems (with regular dynamics) from non-integrable systems (with an irregular and chaotic dynamics). Most of found methods have serious limitations, as they either allow to study integrability only in a very narrow class of functions, or they require that the considered system has certain specific properties which significantly limits their applications. It was only beginning of this century when the effective mathematical tools appeared to allow the study of the existence of first integrals. One of them is the Morales–Ramis theory. It allows to obtain strong, necessary conditions for integrability formulated in the language of differential Galois theory. They can be used both to prove non-integrability or to specify values of parameters for which the considered model is suspected to be integrable.

Due to technical difficulties, the Morales-Ramis theory was applied to special classes of Hamiltonian systems. In the project we propose its extension to n-dimensional systems of ordinary differential equations. In particular, to Hamiltonian systems in multi-dimensional curved spaces. Systems of this type appear in topology, general relativity, cosmology and in quantum mechanics. For instance, most known solutions of the Einstein equation, Schwarzschild and Kerr metrics, have integrable geodesic flow. Systems describing trajectories along geodesics are very important for understanding geometry of manifolds. Recently the geodesic flows have gained a new interest due to their physical meaning as trajectories of massive particles moving on various curved manifolds. Detection of gravity waves is strictly related with the description of the time-evolution of the space- distance between two heavy mass-points on geodesics in the Zipoy-Voorhees metric.

The project is interdisciplinary and the subject is very attractive both from mathematical and physical point of view. Besides the advanced theory of differential equations, special functions, algebraic and differential geometry, we also use numeral methods and techniques. We plan to analyse systems, which are interesting in applications whereas their integrability analysis has not been made so far. The developed methods, formulated theorem and found integrable cases will be interesting to both specialists in the theory of dynamical systems as well as to scientists from various fields of sciences who use systems of ordinary differential equations in their research.