Differential Harnack Inequalities - Nonlinear and Nonlocal

General Audience Summary

Since the Newtonian revolution of the 17th century the scientific description of the physical world has been cast in the language of mathematics. Over the course of the next three centuries it became ever more evident that fundamental laws of nature are most naturally laid out in the form of relations concerning rates of change of a set of basic quantities such as energy, velocity, density or charge. This is the realm of differential equations. It is often relatively straightforward to write down a sensible differential equation for a natural phenomenon. If the equation is particularly simple it may even be solved explicitly. Nature however is rarely so generous and instead calls for notoriously difficult nonlinear partial differential equations, Einstein field equations and Navier-Stokes equations being but two famous examples that are still far from being understood.

In this project we focus on an important class of parabolic equations which are indispensable in many fields of study including the description of heat transfer, fluid flow, chemical reactions, population dynamics, social media or financial markets. Although an attempt to solve such an equation is hopeless in general, mathematicians over decades of intense analytical developments learned how to scrutinise possible solutions without finding them explicitly. We are now able to decide whether a solution exists or not, whether there is one or many possibilities, whether it is nice and smooth or rough and discontinuous, whether it lasts forever or ceases to exist after some time popping out of existence or spectacularly blowing-up.

Among the most useful and sought after pieces of information are so-called differential Harnack estimates that in particular make it possible to compare values of a solution at two different places and at different times. Further consequences of such comparison are manifold including singularity formation and life-span of solutions. Our principal aim is to derive new, sharp differential Harnack inequalities for a class of important nonlinear and nonlocal parabolic equations. In particular for the Fujita equation and for fractional heat equations. Next, in due course we want to extract consequences of these estimates and use them to shed new light on some long-standing open problems concerning existence, regularity and life-span of solutions for the semilinear heat flow. Moreover, we want to explore intriguing connections between differential Harnack inequalities, the optimal transport theory and the thermodynamical concept of entropy.

In the last two decades the optimal transport theory offered a new deep way of interpreting a range of fundamental equations of mathematical physics. The theory of gradient flows with respect to the Wasserstein metric put forward by Jordan, Kinderlehrer and Otto brought home the intuitive picture of evolution of physical systems being driven by the tendency to optimise a particular, physically relevant quantity. However, among many quantities that may change monotonically with evolution some are more useful than other for the purpose of mathematical inquiry and the optimal transport theory helps identify promising candidates and these in turn are reminiscent of the thermodynamical notion of entropy.

The concept of entropy proved particularly powerful in the study of heat related equations. In his famous 1958 work on regularity of parabolic equations John F. Nash wrapped his argument around a quantity associated with a solution that is reminiscent of the Gibbs entropy known from statistical mechanics. More recently Grigorij Perelman applied thermodynamical reasoning to come up with an entropy functional associated with the Ricci flow and this development proved crucial in his resolution of the Poincaré Conjecture. Even though a notion of entropy was central to those important developments it remains mysterious and for many equations it is not at all clear what the *right* definition should be.

Clear connections between differential Harnack inequalities, optimal transport and entropy may be traced for two fundamental models of heat flow: the linear heat equation and the porous medium equation. Respective estimates are called Li-Yau and Aronson-Bénilan inequalities and are known to be the best possible. However, for other fundamental equations such as the Fujita equation or the fractional heat equation it is not clear how to define the relevant entropy functional. In our approach we will take cue from the fact that Li-Yau and Aronson-Bénilan estimates are sharp for very special families of solutions, analogues of which exist for classes of semilinear and nonlocal equations. We will explore how scaling symmetries of equations combined with energy methods may provide guidance in capturing the right form of differential Harnack inequalities.