Analysis of travelling waves in semilinear elliptic problems

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The research project is devoted to study travelling waves to some elliptic problems arising in Bose-Einstein condensates and in nonlinear optics. The physical models are described by the Gross-Pitaevskii equation, the Schrödinger equations and by the electromagnetic wave equation. These models are widely investigated in physics and engineering, however there is a need for an analytical study from the mathematical point of view. Some long-lasting open problems have been answered recently, however the new questions and mathematical challenges appear, which are important from the physical point of view.

Our first aim is to find a large class of nonlinear effects, for which we obtain travelling waves solving Gross-Pitaevskii equation. Moreover we want to investigate the existence of travelling waves in the presence of the external potential, which is used to confine the condensate in the Bose-Einstein model. We intend to study the multiplicity of travelling waves as well.

Secondly, we would like to develop a new concept of multidimensional black and dark solitons in $\mathbb{R}^N$. Next, we also look for two dimensional profiles of the electromagnetic travelling wave fields in nonlinear media described by the Maxwell equations and the material laws. In particular, we want to find profiles taking into account nonlinear effects, possibly sing-changing and involving the quintic nonlinearity.

The project lies at the intersection of the following fields: variational methods, spectral theory, partial differential equations, functional analysis and mathematical physics. In particular we apply and develop the following mathematical tools: spectral theory of elliptic operators, elliptic regularity, strong maximum principle, unique continuation method, Brezis-Kato regularity, mountain pass lemma, linking theory of strongly indefinite problems, dual variational method, minimization techniques on topological manifolds, in particular on Pohozaev-type manifolds, Lusternik-Schnirelmann theory, Krasnoselskii genus and topological degree.

The potential applications of the expected results may result in a better understanding of the physical models, e.g. of nonlinear optics or Bose-Einstein condensates. As a potential output, we expect that new nonlinear phenomena can be treated analytically in the context of travelling waves, a novel concept of dark/black solitons will be developed with the analytical justification, the existence of solutions to our problems and their symmetry properties will be obtained. We plan to develop a functional setting, which allows to study travelling waves by means of PDE methods, where some new mathematical techniques have to be worked out. We are convinced that these new methods will allow to study problems with nonzero conditions at infinity, including strongly indefinite nonlinear partial differential equations like nonlinear wave equations or Schrödinger equations, and will be of interest for specialists in all the above fields.

We expect that the results of the project will give raise to the further studies of the dynamics of the time-dependent Gross-Pitaevskii equation or the nonlinear electromagnetic wave equation