Ulam stability in operator algebras

Popular Science summary

In 1940, Ulam posed a question about the stability of group homomorphisms which can be formulated as follows: Is it true that a mapping between groups which satisfies the homomorphism condition only approximately, must be close to an actual homomorphism? More generally, Ulam's question is formulated by asking under what conditions an object satisfying a certain property approximately must be close to an object satisfying that property exactly.

Throughout the years, studying this type of questions in various contexts led to plenty of interesting applications in Functional Analysis. For example: quasi-linear maps are fundamental for the theory of twisted sums of Banach spaces (Kalton–Peck theory); almost multiplicative maps are useful for understanding the structure and cohomology theory of Banach algebras (B.E. Johnson's theory of the so-called AMNM pairs); approximate *-homomorphisms have been used by Farah to solve the famous problem of inner automorphisms of the Calkin algebra; almost order zero, or almost disjointness preserving operators serve as building blocks for defining the nuclear dimension of C*-algebras (Winter–Zacharias theory).

The project is meant to develop a research where each of the topics listed above would play a role. Our main goal is to extend the list of applications of Ulam stability problems to Functional Analysis, particularly in the noncommutative setting, that is, when considering such questions in operator algebras. The problems included in the project can be divided into four main groups:

(a) the lifting problem for compact perturbations of operator semigroups, studied with the aid of a modification of the classical Brown–Douglas–Fillmore theory, and with applications to investigating spectral properties of Toeplitz operators;

(b) stability of Jordan *-homomorphisms, studied by using approximation methods by B.E. Johnson, adapted to cohomologies of Jordan triples, as well as its consequences in the study of almost order zero maps with possible applications to the theory of nuclear dimension of C*-algebras;

(c) C*-algebraic characterization of the Ulam stability property of group homomorphisms, based on the Brown–Guentner construction of C*-norms which generalizes the classical constructions of group C*-algebras;

(d) noncommutative analogues of the Kalton–Roberts theorem on nearly additive set functions, where we consider (ortho)modular lattices of projections in von Neumann algebras, interplays with Kirchberg's covering number, and possible applications to the theory of extensions of von Neumann algebras.

In all of the above-listed topics, the core is some kind of Ulam's stability problem, like satisfying the semigroup condition modulo the ideal of compact operators, as in (a), or satisfying approximate versions of equations defining relevant properties, as in (b)–(d). The motivation for initiating this project is, first of all, the fact that the posed questions are natural and have potential applications to studying concrete objects like Toeplitz operators or the noncommutative dimension of C*-algebras. An interesting consequence would also be to extend some already existing and important theories like, for example, B.E. Johnson's theory of almost multiplicative operators or the Brown–Guentner theory of C*-completions. Under this project, we will most likely use a wide spectrum of methods such as methods from cohomological algebra, combinatorics or spectral theory.