Normalized solutions to semilinear elliptic problems

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Schrödinger's equation was introduced in 1926 when Erwin Schrödinger derived it from the wave equation. Since then, it has had applications in various fields of Physics: for example it models the behaviour of states in the Bose-Einstein condensation or describes the wave propagation in nonlinear optics.

At the same time, there are several situations where the objects described by solutions to a Schrödinger equation come together with a precise assigned quantity, which we call in general the L^2-*mass*. More precisely, taking into account e.g. the previous contexts, it measures the power supply (nonlinear optics) or the total number of atoms (Bose-Einstein condensation). In addition the L^2-mass is, together with the energy, a conserved quantity in the evolution (i.e. time-dependent) Schrödinger equation.

It is therefore not a surprising fact that solutions to Schrödinger equations with prescribed mass are looked for: such solutions are called *normalized solutions*.

In this research project we investigate some systems of elliptic (i.e. time-independent) Schrödinger equations; such elliptic systems arise when one seeks standing wave solutions to the corresponding evolution systems, i.e. solutions of the form $(x,t) \mapsto e^{-i\lambda t}u(x)$, where λ is the chemical potential. When normalized solutions are involved, such λ's are not given quantities, but on the contrary are part of the unknown: from a mathematical point of view this is the case because they appear as Lagrange multipliers coming from the L^2-mass constraint.

In particular we look for normalized *ground state solutions*. A ground state solution is a nontrivial (i.e. not identically zero) solution which minimizes the energy among the nontrivial solutions: the importance of ground state solutions is due to the fact that natural phenomena occur involving the lowest possible energy, therefore ground states are, in some sense, the only physically reasonable ones.

Since Schödinger's equation models several physical phenomena, we consider autonomous and nonautonomous problems, with different types of nonlinearities and potentials. In autonomous problems, where the potential is identically zero, we focus on nonlinearities in the *infinite mass case*, i.e. when the nonlinearity is strictly sublinear at zero. This allows the energy functional to be infinite-valued and requires an approximation argument. As for nonautonomous problems, we focus on singular potentials which depend on some or all the components of the variable x. We make use of *ad hoc* techniques, e.g. based on nonautonomous Pohožaev-type manifolds, or bring the problems to equivalent autonomous ones.

We expect to find normalized ground state solutions for the problems proposed and, in order to achieve this, we develop new mathematical techniques and/or combine well-known tools from possibly different areas and adapt them to normalized problems.