Rough invariant waves

Mathematicians are often interested in making quantitative predictions about particular phenomena, and the mathematical field of analysis concerns itself with those phenomena that involve movement or change of some sort. A concrete example of this is a form of change that everyone will recognize: dropping an object into a still pool of water. Waves will emanate from where the object hits the water, and the pool will change as the waves progress. Eventually the waves will reach the edge of the pool and reflect back, and it is easy to make quantitative predictions about the trajectory of the waves if the pool is still and its edges are smooth. But what happens when the pool has very many rough edges and corners? One can visualize the chaos that might ensue: waves will reflect off the edges in unexpected ways, subsequently bouncing off each other as well, until it becomes difficult to discern any meaningful structure in the water.

The project is concerned with determining, in a quantitative sense, how rough a pool of water needs to be before it becomes difficult to predict the behavior of waves in that pool. In fact, the project is not only concerned with water waves, it aims to study the behavior of a variety of waves: electromagnetic, sound, radio, even gravitational waves; anything that can be described by what mathematicians call a wave equation. Such equations are satisfied by all the waves that we know from everyday life, and they capture the essential behavior of real-life waves while leaving out the details that only obfuscate the underlying patterns.

More precisely, mathematicians can quantify how smooth a surface is by assigning a number to it. A smooth surface such as a circle will have a very high smoothness number, whereas a surface with lots of rough edges and corners will have a small smoothness number. If the surface is particularly fractal and chaotic, it may even have smoothness number zero. A natural question is then: what is the critical smoothness number below which the behavior of waves on that surface fundamentally changes and it becomes more difficult to make quantitative predictions? This will typically depend on the specific type of behavior under consideration, although for many problems the critical smoothness number has turned out to be two. Recently, in collaboration with others, the principal investigator has shown that for a specific type of wave behavior called Lp-regularity, the critical smoothness number is no larger than two. This means that, on any surface that has smoothness number at least two, one can make the same quantitative predictions about waves that are possible on smooth surfaces without edges or corners.

This project aims to consider the natural follow-up question: for Lp-regularity, is the critical smoothness number two, or is it smaller than two? And if the smoothness number is in fact two, is it then perhaps still possible to make some predictions about waves on very rough surfaces, even if these predictions are not as powerful as for surfaces with smoothness number larger than two?

These questions are important for our understanding of nature and the world around us, since answers to such questions apply to many types of waves on many types of surfaces, using only as input the wave equation and the smoothness number of the surface.

It is expected that the project will determine the critical smoothness number for Lp-regularity, and that it will also lead to a better understanding of waves on surfaces with smoothness number below the critical value.