In everyday life we write and perform operations on numbers using the decimal number system. One can however represent nonnegative integers in base-b number systems for any base $b \geq 2$. In our project we will focus on studying certain interesting properties of such representation. The problems considered lie at the intersection of number theory, combinatorics, and theoretical computer science. The project consists of two closely connected parts:

A Occurrences of values of special block-additive functions in residue classes.

B Variations on Cusick's conjecture.

The first topic is related to a result obtained by Gelfond in 1968. He considered the function s_b computing the sum of digits in the base-b representation of nonnegative integers. He proved that if the numbers $b-1, m$ are relatively prime, then the values of the function s_b evaluated at subsequent terms of any arithmetic progression $(kn+a)_{n\geq 0}$ give the remainders $0, 1, \ldots, m-1$ with equal frequency when divided by m. However, Gelfond's theorem does not tell how long one has to wait for an occurrence of a chosen remainder. This problem was attacked by Morgenbesser, Shallit, and Stoll, who obtained a satisfactory solution for $b = 2, a = 0, m = 2$, that is, the parity of the sum of binary digits of the number kn. This choice of parameters is not accidental, as then the considered value $s_2(kn) \bmod 2$ is equal to the knth term of the famous Thue–Morse sequence. Morgenbesser, Shallit, and Stoll derived an optimal upper bound depending only on k on the minimal n for which the value $s_2(kn)$ is odd. They also obtained a weaker bound in the general case. The objective of our research is the extension of these results to other special *block-additive* functions, characterizing the radix representation of nonnegative integers. First of all, we want to consider an analogous problem arising by replacing the sum of digits with a function counting the occurences of a fixed block of digits in the radix representation of n. In particular, for the block of digits 11 we obtain another well-known sequence, called the Rudin–Shapiro sequence. Experimental computations suggest that we can expect results of the same type as those obtained by Morgenbesser, Shallit, and Stoll. Moreover, we plan to strengthen their result for the sum of digits s_b in the general situation.

The second part of the project is devoted to the variations on a conjecture stated by Cusick in 2012. It says that for any positive integer t the sum of binary digits of $n+t$ is greater or equal to the sum of binary digits of n more than half of the time (with respect to n). Although there exists a simple method to settle the conjecture for each specific value of t, so far it has not been solved in full generality. Only partial results are known, obtained in the works of Drmota, Kauers, Spiegelhofer, Emme, Hubert, Stoll, Wallner and others. The main goal of this part of the project will be the study of problems analogous to Cusick's conjecture, where other interesting functions (including block-additive functions) appear instead of the sum of binary digits. Among others, the original conjecture may be treated as a statement concerning the *finite difference* $s_2(n+t) - s_2(n)$. We plan to study its natural generalization to finite differences of higher order. In particular, preliminary computations suggest that a statement corresponding to Cusick's conjecture for the second-order difference $s_2(n+2t)-2s_2(n+t)+s_2(n)$ should be satisfied for all t. Furthermore, similarly as in the first topic, we are going to consider the problem resulting from replacing the function s_2 by block-counting functions. Yet another modification of Cusick's conjecture that we are interested in, is the question of how often both inequalities $s_2(n+t) \geq s_2(n)$ and $s_2(n+t) \geq s_2(n+2t)$ are satisfied simultaneously, asked by Spiegelhofer and Stoll.