Reg. No: 2020/36/C/ST1/00050; Principal Investigator: dr Michał Jan Miśkiewicz

SINGULARITIES IN NONLINEAR PDES OF GEOMETRIC ORIGIN

MICHAŁ MIŚKIEWICZ

Description for the general public

The starting point of my considerations are geometric objects called *manifolds* – curves, surfaces and higher-dimensional structures. A sphere, a torus (i.e., surface of a donut) and a Möbius band are all examples of important and mathematically interesting 2-dimensional manifolds. A fundamental class of problems in this area is concerned with manifolds of optimal shape (in some sense).

The legend of Carthage indicates that such problems were investigated already in ancient times. According to the tale, Dido was allowed to claim as much land as she could surround with a given strip of bull's hide. In result, she decided the city of Carthage to have circular shape, thus solving the *isoperimetric problem* (finding the figure with a given perimeter and maximal area).

Arguably the most known example of geometric optimization is the classical Plateau's problem. Given a closed curve K, we consider surfaces D whose boundary is K; in this class, we aim to find D with minimal area. Such *minimal surfaces* can be observed in nature, for example as soap films; to see this, one just needs to immerse a wire (playing the role of K) in soap water and take it out.

One could expect such area-minimizing surfaces to be *smooth*; this term has a rather technical definition, but essentially *smooth* means *without corners*. And indeed they are smooth in most part, but if K is twisted enough, different parts of the surface can intersect each other along *singular* arcs and vertices. More precisely, three sheets can form a junction resembling the letter Y (like pages of a book), and six sheets can form a tetrahedral junction. Quite remarkably, these are the only possibilities; a quick internet search can show pictures of soap films with these singularities.

This phenomenon of *singular* points which form a lower-dimensional subset is called *partial regularity*. Although surprizing at first sight, it is quite common in geometric optimization problems. *Harmonic maps*, which are the main focus of my project, are another objects of this kind. Given two manifolds $\mathcal{M}$, $\mathcal{N}$, they are maps from $\mathcal{M}$ to $\mathcal{N}$ which minimize the so called *Dirichlet energy*, and they exhibit the same behavior as minimal surfaces. I am working towards a better understanding of their singularities – how large they are, what they look like at a small scale, and when one can be sure there are no singularities at all.

One of fundamental tools in finding and studying optimal shapes are geometric flows. Consider an example: if one wants to reach the peak of the mountain, the shortest way to do this (although not always safe in practice) is to always follow the direction in which the terrain is the steepest; this is called a *gradient flow*. The equivalent for minimal surfaces is the mean curvature flow – it bends the given surface in order to decrease its area in time, and eventually produce a minimal surface. Another celebrated example is the Ricci flow, responsible for Grigori Perelman's solution of Poincaré conjecture.

For harmonic maps, I aim to study the homogeneous harmonic map flow. It differs from the classical non-homogeneous flow studied by other authors, but it gives new hope for solving some long-standing problems in the theory of harmonic maps.